# HODGE THEORY AND DERIVED CATEGORIES OF CUBIC FOURFOLDS

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Abstract. Cubic fourfolds behave in many ways like K3 surfaces. Certain cubics – conjecturally, the ones that are rational – have specific K3s associated to them geometrically. Hassett has studied cubics with K3s associated to them at the level of Hodge theory, and Kuznetsov has studied cubics with K3s associated to them at the level of derived categories.

These two notions of having an associated K3 should coincide. We prove that they coincide generically: Hassett's cubics form a countable union of irreducible Noether-Lefschetz divisors in moduli space, and we show that Kuznetsov's cubics are a dense subset of these, forming a non-empty, Zariski open subset in each divisor.

## 1. Introduction

It has long been noted that there are remarkable similarities between cubic fourfolds and K3 surfaces [8, 41, 43]. Let $X$ denote a smooth complex cubic hypersurface in $\mathbb{P}^5$. Both the Hodge structure and the derived category of $X$ decompose into some trivial pieces and a K3-like piece.

**Hodge theory.** The Hodge-theoretic viewpoint is due to Hassett [19]. The Hodge diamond of $X$ is

$$\begin{array}{ccccc} & & 1 & & \\ & & 1 & & \\ 0 & 1 & 21 & 1 & 0 \\ & & 1 & & \\ & & 1. & & \end{array}$$

Removing the powers $h^i$ of the hyperplane class, we are left with the primitive cohomology:

$$0 \quad 1 \quad 20 \quad 1 \quad 0.$$

This looks like (a Tate twist of) $H^2$ of a K3 surface $S$, but the intersection forms have different signatures: $(20,2)$ for $H^4_{\mathrm{prim}}(X,\mathbb{Z})$ versus $(19,3)$ for $H^2(S,\mathbb{Z})(-1)$. However, we can often find a codimension-1 sub-Hodge structure of signature $(19,2)$ common to both. For the K3 surface it is $H^2_{\mathrm{prim}}(S,\mathbb{Z})(-1)$, the orthogonal to some ample class $\ell$. For the cubic $X$ it is $\{h^2,T\}^\perp$, the subspace of $H^4_{\mathrm{prim}}(X,\mathbb{Z})$ orthogonal to an integral $(2,2)$-class $T \in H^{2,2}(X,\mathbb{Z})$.

Up to automorphisms of the lattice, both of these situations are governed by a single positive integer $d$. For K3 it is the degree $d = \ell^2$ of the (primitive) ample class $\ell$, while for the cubic fourfold it is the discriminant $d(T)$ of the saturated[1] sublattice generated by $h^2$ and $T$. The cubics possessing such a $T \in H^{2,2}(X, \mathbb{Z})$ form an irreducible divisor $\mathcal{C}_d$ in the 20-dimensional moduli space $\mathcal{C}$ of cubics, non-empty if and only if [19, Thm. 1.0.1]

$$(*) \qquad d > 6 \text{ and } d \equiv 0 \text{ or } 2 \pmod 6.$$

Moreover, $\{h^2, T\}^\perp$ is isomorphic to the primitive cohomology $H^2_{\mathrm{prim}}(S, \mathbb{Z})(-1)$ of some polarised K3 surface $(S, \ell)$ if and only if $d$ satisfies the further condition[2] [19, Thm. 1.0.2]

$$(**) \qquad d \text{ is not divisible by 4, 9, or any odd prime } p \equiv -1 \pmod 3.$$

**Derived category.** The derived category viewpoint is due to Kuznetsov [27]. The line bundles $\mathcal{O}_X$, $\mathcal{O}_X(1)$, and $\mathcal{O}_X(2)$ form an exceptional collection in $D(X)$, and the right orthogonal

$$\begin{aligned} \mathcal{A}_X &:= \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^\perp \\ &:= \{E \in D(X) \colon R\operatorname{Hom}(\mathcal{O}_X(i), E) = 0 \text{ for } i = 0, 1, 2\} \end{aligned}$$

looks like the derived category of a K3 surface in that it has the same Serre functor and Hochschild (co)homology. In general, $\mathcal{A}_X$ should be thought of as a non-commutative K3 surface: it is a deformation of the derived category of a genuine K3 surface but, for general $X$, we will see it lacks point-like objects – that is, objects $\kappa \in \mathcal{A}_X$ with $\operatorname{Ext}^*_{\mathcal{A}_X}(\kappa, \kappa) \cong \operatorname{Ext}^*_{\mathrm{K3}}(\mathcal{O}_{\mathrm{point}}, \mathcal{O}_{\mathrm{point}})$. We will call $\mathcal{A}_X$ *geometric* if $\mathcal{A}_X \cong D(S)$ for some projective[3] K3 surface $S$.

**Rationality.** Although it is not strictly relevant to our paper, a strong motivation is the question of which cubic fourfolds are rational.

Harris and Hassett considered the possibility that $X$ is rational if and only if $X$ has an associated K3 in the Hodge-theoretic sense: i.e. $X \in \mathcal{C}_d$ for some $d$ satisfying $(**)$. Hassett [18] found many examples of rational cubics satisfying this condition, although it should be said that no cubic fourfold has yet been shown to be irrational, and Harris and Hassett were always very cautious about conjecturing that $(**)$ should be equivalent to rationality.

Kuznetsov [27] conjectured that $X$ is rational if and only if $X$ has an associated K3 in the derived category sense: i.e. $\mathcal{A}_X$ is geometric. He showed

[1] The discriminant of a lattice is the determinant of the pairing matrix in a basis, so $d(T) = \det \begin{pmatrix} h^2 & h.T \\ h.T & T^2 \end{pmatrix} = 3T^2 - (h.T)^2$. A sublattice $M$ of a lattice $L$ is called *saturated* or *primitive* if $L/M$ is torsion-free, or, equivalently, if $M$ equals its *saturation* $M^{\perp\perp}$. By changing $T$ if necessary, we will always assume that $\langle h^2, T \rangle$ is saturated.

[2] For $d$ even (which is implied by $(*)$), this strange-looking numerical condition turns out to be equivalent to $d$ being the norm of a primitive vector in the $A_2$ lattice $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

[3] "Projective" may be redundant: $\mathcal{A}_X$ is *saturated* in the sense of [10], and it is expected that if $S$ is not projective then $D(S)$ is not saturated [10, Rem. 5.6.2].

that this is true of the known rational cubics. His conjecture has attracted a great deal of interest recently [34, 6, 4].

1.1. **Our results.** At the very least, Hassett and Kuznetsov's conditions should be the same[4], which is what we prove for the generic K3.

**Theorem 1.1.** *If $\mathcal{A}_X$ is geometric then $X \in \mathcal{C}_d$ for some $d$ satisfying $(*)$ and $(**)$. Conversely, for each $d$ satisfying $(*)$ and $(**)$, the set of cubics $X \in \mathcal{C}_d$ for which $\mathcal{A}_X$ is geometric is non-empty and Zariski open.*

We mention again that $\mathcal{C}_d$ is irreducible, so this Zariski open set is dense. Of course we fully expect it is all of $\mathcal{C}_d$, and discuss some of the technical difficulties in trying to prove this in Section 7.3.

As an application we find that certain Hodge cycles on products of cubics and K3s are algebraic. Given a cubic fourfold $X \in \mathcal{C}_d$ with special class $T \in H^{2,2}(X,\mathbb{Z})$ and associated polarised K3 surface $S$, Hassett shows that $\{h^2, T\}^\perp$ and $H^2_{\mathrm{prim}}(S,\mathbb{Z})(-1)$ are isomorphic as abstract lattices. Our results imply that in fact there is a rational cycle on $X \times S$ inducing the isomorphism. Indeed for the dense set of $X \in \mathcal{C}_d$ in Theorem 1.1 this cycle is the Mukai vector of our Fourier-Mukai kernel. (Proposition 5.2 shows that this has the correct cohomology class.) Passing to the closure of this dense set gives the result for all $X \in \mathcal{C}_d$.

1.2. **Strategy.** An outline of the paper is as follows. In Section 2 we define a Mukai lattice for the category $\mathcal{A}_X$, that is, a weight-2 Hodge structure which is isomorphic to the usual Mukai lattice $H^*(S,\mathbb{Z})$ when $\mathcal{A}_X \cong D(S)$. We do this using topological K-theory; in fact most of the lattice-theoretic complications in this subject stem from the difference between the integral structures on cohomology and K-theory.

We spend some time relating this Mukai lattice to $H^4(X,\mathbb{Z})$. In Section 3 this enables us to interpret $(**)$ in much more natural K-theoretic terms. Namely, $X \in \mathcal{C}_d$ for some $d$ satisfying $(**)$ if and only if there are classes $\kappa_1, \kappa_2 \in K_{\mathrm{alg}}(\mathcal{A}_X)$ that behave like the classes of a skyscraper sheaf and ideal sheaf of a point on a K3 surface: $\chi(\kappa_1,\kappa_1) = \chi(\kappa_2,\kappa_2) = 0$ and $\chi(\kappa_1,\kappa_2) = 1$.

Morally, this is the reason for the equivalence of Hassett's and Kuznetsov's conditions. We should now "just" produce the K3 surface $S$ as a moduli space of objects in $\mathcal{A}_X$ of class $\kappa_1 \in K_{\mathrm{top}}(\mathcal{A}_X)$. To do this we first work in $\mathcal{C}_d \cap \mathcal{C}_8$, then use deformation theory to reach a Zariski open subset of $\mathcal{C}_d$.

The advantage of $\mathcal{C}_8$ is that we have Kuznetsov's description of $\mathcal{A}_X$ for $X \in \mathcal{C}_8$ as the derived category of *twisted*[5] sheaves on a K3 surface $S$. So in Section 4 we first we use some lattice theory to show that $\mathcal{C}_d \cap \mathcal{C}_8$ is non-empty

[4]This should be thought of as a non-commutative extension of the result of Mukai and Orlov that two algebraic $K3$ surfaces have equivalent derived categories if and only if they have isomorphic Mukai lattices.

[5]Notice that this does not make $\mathcal{A}_X$ geometric unless the twisting cocycle vanishes, and indeed $d = 8$ does not satisfy $(**)$.

and contains cubics $X$ whose extra discriminant-$d$ class forces the twisting cocyle to vanish. Therefore $\mathcal{A}_X \cong D(S)$ is indeed geometric. However, this equivalence is the "wrong" one, expressing $S$ as a moduli space of objects in class different from $\kappa_1$, so it does not deform out of $\mathcal{C}_8$.

But we have still gained something: since $\mathcal{A}_X$ is geometric we can now work in $D(S)$, where the powerful results of Mukai [37] give us a moduli space of stable sheaves on $S$ whose class in $K_{\mathrm{top}}(\mathcal{A}_X)$ is $\kappa_1$. Replacing $S$ by this moduli space in Section 5 gives us the "correct" equivalence $\mathcal{A}_X \cong D(S)$.

In Section 7 we deform $X$ inside $\mathcal{C}_d$. There is a corresponding deformation of $S$, using the Torelli theorem and the close relationship between the cohomologies of $X$ and $S$. That their Hodge structures remain related (in the sense of Hassett) in the family means that the cohomological obstruction to deforming the equivalence – or, more precisely, its Fourier-Mukai kernel – vanishes. That is, the Mukai vector of the kernel remains of type $(p,p)$ in the family. Using some Hochschild (co)homology theory set up in Section 6 we show that this implies the vanishing of the obstructions to deforming the kernel to any order. By algebraicity, the equivalence deforms over a Zariski open subset of $\mathcal{C}_d$.

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**Notation.** All our varieties are smooth and complex projective. We usually use $X$ to denote a cubic 4-fold, and $S$ a K3 surface. Both have torsion-free cohomology, so it makes sense to let $H^{p,p}(X,\mathbb{Z})$ denote the intersection $H^{p,p}(X) \cap H^{2p}(X,\mathbb{Z})$ inside $H^{2p}(X,\mathbb{C})$.

$D(Y)$ denotes the bounded derived category of coherent sheaves. In Section 7.2, where $Y$ might denote a smooth family over an Artinian base, it denotes the bounded derived category of perfect complexes.

The Mukai vector of $E \in D(Y)$ is

$$v(E) = \mathrm{ch}(E)\,\mathrm{td}(Y)^{1/2} \ \in\ H^*(Y,\mathbb{Q}). \tag{1.1}$$

Any object $P \in D(Y \times Z)$ induces a Fourier-Mukai functor $\Phi_P \colon D(Y) \to D(Z)$ defined by the formula

$$\Phi_P(\,\cdot\,) := \pi_{Z*}(\pi_Y^*(\,\cdot\,) \otimes P),$$

where all functors are derived unless otherwise stated. There is an induced map $\Phi_P^{H^*}$ on rational cohomology given by

$$\Phi_P^{H^*}(\,\cdot\,) := \pi_{Z*}\big(\pi_Y^*(\,\cdot\,) \cup v(P)\big) \colon\ H^*(Y,\mathbb{Q}) \longrightarrow H^*(Z,\mathbb{Q}). \tag{1.2}$$

## 2. A Mukai lattice for $\mathcal{A}_X$

In this section we introduce a weight-2 Hodge structure which we call the *Mukai lattice* of $\mathcal{A}_X$, and relate it to the lattice $H^4(X,\mathbb{Z})$.

2.1. **The Mukai lattice of a K3 surface.** Recall that the Mukai lattice of a K3 surface $S$ consists of the Abelian group

$$H^*(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z}),$$

endowed with the Mukai pairing[6]

$$v.w := v_0 w_4 - v_2 w_2 + v_4 w_0$$

and the weight-2 Hodge structure

$$\begin{aligned}\widetilde{H}^{2,0} &:= H^{2,0}(S),\\ \widetilde{H}^{1,1} &:= H^{0,0}(S) \oplus H^{1,1}(S) \oplus H^{2,2}(S),\\ \widetilde{H}^{0,2} &:= H^{0,2}(S),\end{aligned}$$

on $H^*(S,\mathbb{Z}) \otimes \mathbb{C}$. If $E \in D^b(S)$ then the Mukai vector $v(E) \in H^*(S,\mathbb{Q})$ (1.1) actually lies in $H^*(S,\mathbb{Z})$ and satisfies

$$v(E_1).v(E_2) = \chi(E_1,E_2).$$

The algebraic lattice $H^*(S,\mathbb{Z}) \cap \widetilde{H}^{1,1}$ is $-\operatorname{Pic}(S) \oplus U$, where $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the hyperbolic plane spanned by $v(\mathcal{O}_{\text{point}})$ and $v(\mathcal{I}_{\text{point}})$.

Any equivalence $D(S_1) \cong D(S_2)$ induces a Hodge isometry $H^*(S_1,\mathbb{Z}) \cong H^*(S_2,\mathbb{Z})$. That it preserves the *integral* structure is special to K3 surfaces; usually Fourier-Mukai functors only induce maps on rational cohomology (1.2). But there is another integral structure on rational cohomology that they *do* respect: topological K-theory (or rather its image under the Mukai vector).

2.2. **Review of topological K-theory.** The use of topological K-theory in algebraic geometry is quite rare[7] so we give a brief review, using [17] as our main reference. We take

$$K_{\text{top}}(X) := K^0_{\text{top}}(X) \oplus K^1_{\text{top}}(X),$$

although for the spaces we consider $K^1_{\text{top}}$ vanishes. The Mukai vector induces an isomorphism of vector spaces $K_{\text{top}}(X) \otimes \mathbb{Q} \to H^*(X,\mathbb{Q})$. A holomorphic map $f : X \to Y$ induces, in addition to the usual pullback map $f^* : K_{\text{top}}(Y) \to K_{\text{top}}(X)$, a pushforward map $f_* : K_{\text{top}}(X) \to K_{\text{top}}(Y)$ satisfying a projection formula and a Grothendieck-Riemann-Roch formula, and compatible with the pushforward on $K_{\text{alg}}$ [3]. Applying this to the projection $X \times Y \to Y$ we find that a Fourier-Mukai functor $\Phi_P : D(X) \to D(Y)$

[6] Mukai uses the opposite sign for this pairing so that $H^2(S,\mathbb{Z})$ is a sublattice of $H^*(S,\mathbb{Z})$ and $v(E_1).v(E_2) = -\chi(E_1,E_2)$. In our convention, $-H^2(S,\mathbb{Z})$ is a sublattice and $v(E_1).v(E_2) = \chi(E_1,E_2)$.

[7] But see [20, Rmk. 3.4].

induces a map $\Phi_P^K : K_{\rm top}(X) \to K_{\rm top}(Y)$ compatible with the usual induced maps:

$$\begin{tikzcd} D(X) \arrow[r] \arrow[d, "\Phi_P"'] & K_{\rm alg}(X) \arrow[r] \arrow[d, "\Phi_P^K"'] & K_{\rm top}(X) \arrow[r, "v"] \arrow[d, "\Phi_P^K"'] & H^*(X,\mathbb{Q}) \arrow[d, "\Phi_P^{H^*}"'] \\ D(X) \arrow[r] & K_{\rm alg}(X) \arrow[r] & K_{\rm top}(X) \arrow[r, "v"] & H^*(X,\mathbb{Q}). \end{tikzcd}$$

We also have Grothendieck-Verdier duality[8]

$$f_*(\kappa^\vee \cdot (-1)^{\dim X}[\omega_X]) = f_*(\kappa)^\vee \cdot (-1)^{\dim Y}[\omega_Y]$$

and the Euler pairing

$$\kappa_1 . \kappa_2 := p_*({\kappa_1}^\vee \cdot \kappa_2) \,\in\, K_{\rm top}(\text{point}) = \mathbb{Z}$$

on $K_{\rm top}(X)$, where $p : X \to \text{point}$.

Putting these together with the fact that

$$\Psi := \Phi_{P^\vee \otimes \omega_Y[\dim Y]} : D(Y) \longrightarrow D(X)$$

is the right adjoint to $\Phi_P$ we see that the induced map $\Psi^K$ on K-theory is right adjoint to $\Phi_P^K$.

2.3. **The Mukai lattice of a K3 surface revisited.** If $S$ is a K3 surface then $v : K_{\rm top}(S) \to H^*(S,\mathbb{Z})$ is an isomorphism, so we can recast the Mukai lattice of $S$ as follows. It consists of the Abelian group $K_{\rm top}(S)$ endowed with the Mukai pairing

$$\kappa_1 . \kappa_2 := \chi(\kappa_1, \kappa_2)$$

and the weight-2 Hodge structure pulled back by the complexified Mukai vector $v : K_{\rm top}(S) \otimes \mathbb{C} \xrightarrow{\sim} H^*(S,\mathbb{C})$:

$$\begin{aligned} \widetilde{H}^{2,0} &:= v^{-1}(H^{2,0}(S)), \\ \widetilde{H}^{1,1} &:= v^{-1}(H^{0,0}(S) \oplus H^{1,1}(S) \oplus H^{2,2}(S)), \\ \widetilde{H}^{0,2} &:= v^{-1}(H^{0,2}(S)). \end{aligned}$$

Since the integral Hodge conjecture holds for surfaces, the algebraic lattice $K_{\rm top}(S) \cap \widetilde{H}^{1,1}$ in this Hodge structure equals the numerical K-theory

$$K_{\rm num}(S) \,:=\, K_{\rm alg}(S)/\ker\chi \,=\, \mathrm{im}(K_{\rm alg}(S) \to K_{\rm top}(S)).$$

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[8]This follows from the Riemann-Roch theorem in the form [17, Thm. 10] (modulo a misprint) applied to the multiplicative transformation $(-)^\vee$, which topologists call the Adams operation $\psi^{-1}$, and the fact that for a vector bundle $\pi : E \to X$, the Thom class $U \in K^{2\,\mathrm{rank}\,E}(E, E \setminus \text{zero section})$ satisfies $U^\vee = U \cdot \pi^*[\det E]$. There is a small subtlety in that $f_*$ naturally goes from $K^i(X)$ to $K^{i-2\dim f}(Y)$, and the Bott isomorphism $\beta : K^j(Y) \to K^{j+2}(Y)$ has $\beta(\kappa)^\vee = -\beta(\kappa^\vee)$.

2.4. **A Mukai lattice for $\mathcal{A}_X$.** The foregoing discussion suggests the following definition.

**Definition 2.1.** If $X$ is a cubic fourfold, the *Mukai lattice of* $\mathcal{A}_X$ consists of the Abelian group

$$K_{\mathrm{top}}(\mathcal{A}_X) := \{\kappa \in K_{\mathrm{top}}(X) : \chi([\mathcal{O}_X(i)], \kappa) = 0 \text{ for } i = 0, 1, 2\}$$

with the Mukai pairing

$$\kappa_1 . \kappa_2 := \chi(\kappa_1, \kappa_2) \tag{2.1}$$

and the weight-2 Hodge structure pulled back via the complexified Mukai vector $v : K_{\mathrm{top}}(\mathcal{A}_X) \otimes \mathbb{C} \hookrightarrow H^*(X, \mathbb{C})$:

$$\begin{aligned}
\widetilde{H}^{2,0} &= v^{-1}(H^{3,1}(X)),\\
\widetilde{H}^{1,1} &= v^{-1}(H^{0,0}(X) \oplus H^{1,1}(X) \oplus H^{2,2}(X) \oplus H^{3,3}(X) \oplus H^{4,4}(X)),\\
\widetilde{H}^{0,2} &= v^{-1}(H^{1,3}(X)).
\end{aligned}$$

We will see in Proposition 2.4 that the algebraic lattice $K_{\mathrm{top}}(\mathcal{A}_X) \cap \widetilde{H}^{1,1}$ is

$$K_{\mathrm{num}}(\mathcal{A}_X) \;:=\; K_{\mathrm{alg}}(\mathcal{A}_X)/\ker\chi \;=\; \mathrm{im}(K_{\mathrm{alg}}(\mathcal{A}_X) \to K_{\mathrm{top}}(X)).$$

If $S$ is a K3 surface and $\Phi : D(S) \to D(X)$ is a fully faithful functor with image $\mathcal{A}_X$ we get a Hodge isometry

$$\Phi^K : K_{\mathrm{top}}(S) \xrightarrow{\sim} K_{\mathrm{top}}(\mathcal{A}_X)$$

with inverse induced by the right adjoint of $\Phi$. Since there are cubics $X$ with $\mathcal{A}_X \cong D(S)$ (and all smooth cubics are deformation equivalent), the Mukai pairing (2.1) is symmetric, which was not obvious *a priori*. It is abstractly isomorphic to the even unimodular[9] lattice

$$U^{\oplus 4} \oplus E_8{}^{\oplus 2}.$$

2.5. **Relation to $H^4(X, \mathbb{Z})$.** We begin with a powerful topological fact.

**Proposition 2.2.**

- *$K_{\mathrm{top}}(X)$ is torsion-free, so $v : K_{\mathrm{top}}(X) \to H^*(X, \mathbb{Q})$ is injective.*
- *For any $\kappa \in K_{\mathrm{top}}(X)$, the leading term of $v(\kappa)$ is integral: writing $v(\kappa) = v_j + v_{j+1} + \cdots$ with $v_i \in H^i(X, \mathbb{Q})$, we have $v_j \in H^j(X, \mathbb{Z})$.*
- *For any $j$ and any $v_j \in H^j(X, \mathbb{Z})$ there is a $\kappa \in K_{\mathrm{top}}(X)$ with $v(\kappa) = v_j +$ higher-degree terms.*

*Proof.* By the Lefschetz hyperplane theorem and Poincaré duality, $H^*(X, \mathbb{Z})$ is torsion free, so the Atiyah-Hirzebruch spectral sequence degenerates at the $E_2$ page. Then the claims are standard consequences of this covered in [17], or more clearly in [21, p. 73ff]. □

[9]A lattice is called *unimodular* if its discriminant is $\pm 1$, i.e. the natural map $L \to L^*$ is an isomorphism.

The numerical Grothendieck group $K_{\mathrm{num}}(\mathcal{A}_X)$ always contains at least two classes, given by projecting[10] $\mathcal{O}_{\text{line}}$ and $\mathcal{O}_{\text{point}}$ into $\mathcal{A}_X$. In fact we will find it more convenient to use $\mathcal{O}_{\text{line}}(1)$ and $\mathcal{O}_{\text{line}}(2)$ instead. Precisely, let

$$\begin{aligned} \mathrm{pr}_i : K_{\mathrm{top}}(X) &\to K_{\mathrm{top}}(X), \\ \kappa &\mapsto \kappa - \chi([\mathcal{O}_X(i)], \kappa) \cdot [\mathcal{O}_X(i)], \end{aligned}$$

be the projection onto $[\mathcal{O}_X(i)]^\perp$, i.e. the map induced by left mutation past $\mathcal{O}_X(i)$. Then

$$\mathrm{pr} := \mathrm{pr}_0 \circ \mathrm{pr}_1 \circ \mathrm{pr}_2$$

projects $K_{\mathrm{top}}(X)$ onto $K_{\mathrm{top}}(\mathcal{A}_X)$ and is left adjoint to the inclusion $K_{\mathrm{top}}(\mathcal{A}_X) \hookrightarrow K_{\mathrm{top}}(X)$. Now define

$$\lambda_1 := \mathrm{pr}([\mathcal{O}_{\text{line}}(1)]) \text{ and } \lambda_2 := \mathrm{pr}([\mathcal{O}_{\text{line}}(2)]).$$

Calculating their Euler pairing we find they generate the negative definite sublattice

$$-A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \subset K_{\mathrm{top}}(\mathcal{A}_X). \tag{2.2}$$

The orthogonal to these classes in $K_{\mathrm{top}}(\mathcal{A}_X)$ is comparable to $H^4_{\mathrm{prim}}(X, \mathbb{Z})$: in the first case we have removed $[\mathcal{O}_X(i)]$, $i = 0, 1, \ldots, 4$, from $K_{\mathrm{top}}(X)$ while in the second case we have removed $h^i$, $i = 0, 1, \ldots, 4$, from $H^*(X, \mathbb{Z})$. Remarkably, they are in fact integrally isometric via the Mukai vector.

**Proposition 2.3.** *The Mukai vector $v : K_{\mathrm{top}}(\mathcal{A}_X) \to H^*(X, \mathbb{Q})$ takes $\{\lambda_1, \lambda_2\}^\perp \subset K_{\mathrm{top}}(\mathcal{A}_X)$ isometrically onto $\{h^2\}^\perp \subset H^4(X, \mathbb{Z})$ :*

$$\{\lambda_1, \lambda_2\}^\perp \;\cong\; H^4_{\mathrm{prim}}(X, \mathbb{Z}). \tag{2.3}$$

*More generally, if $\kappa_1, \ldots, \kappa_n \in K_{\mathrm{top}}(\mathcal{A}_X)$ then the Mukai vector takes $\{\lambda_1, \lambda_2, \kappa_1, \ldots, \kappa_n\}^\perp$ isometrically onto $\{h^2, c_2(\kappa_1), \ldots, c_2(\kappa_n)\}^\perp$.*

*Proof.* Since the cohomology of $X$ is so simple,

$$H^*(X, \mathbb{Q}) \cong \langle 1, h, h^2, h^3, h^4 \rangle \oplus H^4_{\mathrm{prim}}(X, \mathbb{Q}),$$

and the Todd class of $X$ is a linear combination of the $h^i$, we find that for any $\kappa \in K_{\mathrm{top}}(X)$,

$$\begin{aligned} &\kappa \in \{\lambda_1, \lambda_2\}^\perp \subset K_{\mathrm{top}}(\mathcal{A}_X) \\ \iff\ &\kappa \in \{\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \lambda_1, \lambda_2\}^\perp \subset K_{\mathrm{top}}(X) \\ \iff\ &\kappa \in \{\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \mathcal{O}_X(3), \mathcal{O}_X(4)\}^\perp \subset K_{\mathrm{top}}(X) \\ \iff\ &v(\kappa) \in \{1, h, h^2, h^3, h^4\}^\perp \subset H^*(X, \mathbb{Q}) \\ \iff\ &v(\kappa) \in \{h^2\}^\perp \subset H^4(X, \mathbb{Q}) \\ \iff\ &v(\kappa) \in \{h^2\}^\perp \subset H^4(X, \mathbb{Z}), \end{aligned}$$

[10] Recall that cubics contain many lines in $\mathbb{P}^5$. Of course for $\mathcal{A}_X$ to be $D(S)$ we need at least *three* classes in $K_{\mathrm{num}}(\mathcal{A}_X)$, corresponding to the classes in $K_{\mathrm{num}}(S)$ of $\mathcal{O}_{\text{point}}$, $\mathcal{I}_{\text{point}}$ and a polarisation $L$.

where the last line comes from the leading term of $v(\kappa)$ being integral.

So we must show that any $T \in \{h^2\}^\perp \subset H^4(X,\mathbb{Z})$ is the image of some $\tau \in \{\lambda_1,\lambda_2\}^\perp \subset K_{\mathrm{top}}(\mathcal{A}_X)$. By Proposition 2.2 there is a class $\tau' \in K_{\mathrm{top}}(X)$ with $v(\tau') = T +$ higher-degree terms. Since $T.h^2 = 0$, the Hirzebruch-Riemann-Roch formula gives

$$\chi(\tau' \cdot \mathcal{O}_X(t)) = at + b \quad \text{for some } \ a, b \in \mathbb{Z}.$$

Set $\tau := \tau' - a[\mathcal{O}_{\mathrm{line}}] + (a-b)[\mathcal{O}_{\mathrm{point}}]$. Then $\chi(\tau \cdot \mathcal{O}_X(t)) = 0$ for all $t$, so $\tau \in \{\lambda_1,\lambda_2\}^\perp \subset K_{\mathrm{top}}(\mathcal{A}_X)$.

The more general statement follows from the identity

$$\chi(\tau,\kappa_i) = c_2(\tau)\cdot c_2(\kappa_i) \quad \text{for any } \tau \in \{\lambda_1,\lambda_2\}^\perp \subset K_{\mathrm{top}}(\mathcal{A}_X),$$

which is Hirzebruch-Riemann-Roch coupled with the fact we just proved that $v(\tau)$ lies in $H^4_{\mathrm{prim}}(X,\mathbb{Z})$ (and so equals its leading term $-c_2(\tau)$). □

For a unimodular lattice $L$ and a non-degenerate sublattice $M$, the obvious inclusion $M^\perp \subset L/M$ is not an isomorphism unless $M$ is unimodular. Thus $\{\lambda_1,\lambda_2\}^\perp \subsetneq K_{\mathrm{top}}(\mathcal{A}_X)/\langle\lambda_1,\lambda_2\rangle$ and $\{h^2\}^\perp \subsetneq H^4(X,\mathbb{Z})/\langle h^2\rangle$. But the isomorphism (2.3) of Proposition 2.3 extends to the bigger groups too:

**Proposition 2.4.** *The second Chern class $c_2$ descends to an isomorphism of abelian groups*

$$\bar{c}_2\colon \frac{K_{\mathrm{top}}(\mathcal{A}_X)}{\langle\lambda_1,\lambda_2\rangle} \xrightarrow{\ \sim\ } \frac{H^4(X,\mathbb{Z})}{\langle h^2\rangle}.$$

*The preimage of $H^{2,2}(X,\mathbb{Z})/\langle h^2\rangle$ is the image of $K_{\mathrm{alg}}(\mathcal{A}_X)$. In particular, $K_{\mathrm{top}}(\mathcal{A}_X) \cap \widetilde{H}^{1,1}(X) = K_{\mathrm{num}}(\mathcal{A}_X)$.*

*Proof.* Taking $c_2$ and projecting gives a map $K_{\mathrm{top}}(\mathcal{A}_X) \to H^4(X,\mathbb{Z})/\langle h^2\rangle$. This is a group homomorphism because

$$c_2(\kappa_1+\kappa_2) = c_2(\kappa_1) + c_1(\kappa_1)c_1(\kappa_2) + c_2(\kappa_2)$$

and the middle term lies in $\langle h^2\rangle$.

Clearly $\langle\lambda_1,\lambda_2\rangle$ lies in the kernel. Conversely, take $\kappa \in K_{\mathrm{top}}(\mathcal{A}_X)$ with $c_2(\kappa)$ a multiple of $h^2$. Pick two distinct smooth hyperplane sections $H_1, H_2$ of $X$. The leading terms of the Mukai vectors of

$$[\mathcal{O}_X], [\mathcal{O}_{H_1}], [\mathcal{O}_{H_1\cap H_2}], [\mathcal{O}_{\mathrm{line}}], [\mathcal{O}_{\mathrm{point}}] \tag{2.4}$$

are the $\mathbb{Z}$-generators of $\langle 1,h,h^2,h^3,h^4\rangle_{\mathbb{Q}} \cap H^4(X,\mathbb{Z})$:

$$1,\ h,\ h^2,\ h^3/3,\ h^4/3$$

respectively. Therefore by an induction on the degree of the leading term of $v(\kappa)$ – which is always integral by Proposition 2.2 – we find that $\kappa$ is in the integral span of (2.4). Applying pr to the classes (2.4) gives

$$0,\ 0,\ 0,\ 2\lambda_1-\lambda_2,\ \lambda_2-\lambda_1,$$

so $\kappa = \mathrm{pr}(\kappa)$ is a linear combination of $\lambda_1,\lambda_2$.

Thus $\bar{c}_2$ is injective. To see that it is surjective, let $T \in H^4(X, \mathbb{Z})$. By Proposition 2.2, there is a $\tau \in K_{\mathrm{top}}(X)$ with $v(\tau) = -T +$ higher-degree terms. In particular, $c_2(\tau) = T$.

Then $\mathrm{pr}(\tau) \in \mathcal{A}_X$ differs from $\tau$ by a linear combination of $[\mathcal{O}_X]$, $[\mathcal{O}_X(1)]$, and $[\mathcal{O}_X(2)]$ whose Chern classes are all multiples of $h^i$. Therefore $c_2(\mathrm{pr}(\tau))$ differs from $c_2(\tau) = T$ by a multiple of $h^2$.

If $T \in H^{2,2}(X, \mathbb{Z})$ we can take $\tau$ above to be the image of an algebraic class, since Voisin has proved the *integral* Hodge conjecture for cubic 4-folds [44]. Therefore the natural map $K_{\mathrm{alg}}(\mathcal{A}_X) \to K_{\mathrm{top}}(\mathcal{A}_X) \cap \widetilde{H}^{1,1}(X)$ is surjective. It factors through an injection $K_{\mathrm{num}}(\mathcal{A}_X) \hookrightarrow K_{\mathrm{top}}(\mathcal{A}_X) \cap \widetilde{H}^{1,1}(X)$, again since the Hodge conjecture holds for $X$. □

**Proposition 2.5.** *Given $\kappa_1, \dots, \kappa_n \in K_{\mathrm{top}}(\mathcal{A}_X)$, consider the lattices*

$$M_H := \langle h^2, c_2(\kappa_1), \dots, c_2(\kappa_n) \rangle \subset H^4(X, \mathbb{Z}),$$
$$M_K := \langle \lambda_1, \lambda_2, \kappa_1, \dots, \kappa_n \rangle \subset K_{\mathrm{top}}(\mathcal{A}_X).$$

*A class $\kappa \in K_{\mathrm{top}}(\mathcal{A}_X)$ is in $M_K$ if and only if $c_2(\kappa) \in M_H$. The two lattices have the same discriminant; if the signature of $M_H$ is $(r, s)$ then the signature of $M_K$ is $(r-1, s+2)$; and $M_H$ is saturated if and only if $M_K$ is.*

*Proof.* We use the isomorphism $\bar{c}_2$ of Proposition 2.4. Under the two projections

$$M_K \subset K_{\mathrm{top}}(\mathcal{A}_X) \xrightarrow{\pi_K} \frac{K_{\mathrm{top}}(\mathcal{A}_X)}{\langle \lambda_1, \lambda_2 \rangle} \cong \frac{H^4(X, \mathbb{Z})}{\langle h^2 \rangle} \xleftarrow{\pi_H} H^4(X, \mathbb{Z}) \supset M_H$$

the sublattices $M_K$ and $M_H$ project to the same subgroup

$$\overline{M} := \langle [\kappa_1], \dots, [\kappa_n] \rangle \;\cong\; \langle \bar{c}_2(\kappa_1), \dots, \bar{c}_2(\kappa_n) \rangle.$$

Moreover $M_K$ and $M_H$ contain the kernels of the projections, so in fact

$$M_K = \pi_K^{-1}(\overline{M}) \quad \text{and} \quad M_H = \pi_H^{-1}(\overline{M}), \tag{2.5}$$

from which it follows that for any $\kappa \in K_{\mathrm{top}}(\mathcal{A}_X)$,

$$\kappa \in M_K \iff \bar{c}_2(\kappa) \in \overline{M} \iff c_2(\kappa) \in M_H.$$

Similarly (2.5) implies that

$$M_K \text{ is saturated} \iff \overline{M} \text{ is saturated} \iff M_H \text{ is saturated},$$

and more generally

$$i(M_K) = i(\overline{M}) = i(M_H), \tag{2.6}$$

where for any sublattice $M$ we let $i(M)$ denote the index of $M$ in its saturation $M^{\perp\perp}$.

Next, $H^4(X, \mathbb{Z})$ has signature $(21, 2)$ by the Hodge-Riemann bilinear relations, and $K_{\mathrm{top}}(\mathcal{A}_X) \cong U^{\oplus 4} \oplus E_8^{\oplus 2}$ has signature $(20, 4)$. Therefore, using the isometry $M_K^{\perp} \cong M_H^{\perp}$ of Proposition 2.3,

$$\mathrm{sig}\, M_H = (r, s) \iff \mathrm{sig}(M_H^{\perp} = M_K^{\perp}) = (21-r, 2-s) \iff \mathrm{sig}\, M_K = (r-1, s+2).$$

In particular, the discriminants of $M_H$ and $M_K$ have the same sign. Now, both $K_{\mathrm{top}}(\mathcal{A}_X)\cong U^{\oplus 4}\oplus E_8{}^{\oplus 2}$ and $H^4(X,\mathbb{Z})$ are unimodular; the latter by Poincaré duality. Therefore, letting $i$ denote any of the indices (2.6),

$$\begin{aligned}|\operatorname{disc} M_K| &= i^2|\operatorname{disc} M_K^{\perp\perp}|\\ &= i^2|\operatorname{disc}(M_K^{\perp}=M_H^{\perp})| = i^2|\operatorname{disc} M_H^{\perp\perp}| = |\operatorname{disc} M_H|\end{aligned}$$

by [38, Cor. 1.6.2] □

## 3. Interpretation of the numerical condition

If $\mathcal{A}_X\cong D(S)$ for some K3 surface $S$ then the classes $\kappa_1=[\mathcal{O}_{\mathrm{point}}]$ and $\kappa_2=[\mathcal{I}_{\mathrm{point}}]$ span a copy of $U=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ in $K_{\mathrm{num}}(\mathcal{A}_X)$. Combined with the following result this proves the easier direction of Theorem 1.1.

**Theorem 3.1.** *The cubic $X$ lies in $\mathcal{C}_d$ for some $d$ satisfying (∗∗) if and only if the lattice $K_{\mathrm{num}}(\mathcal{A}_X)$ contains a copy of the hyperbolic plane $U=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.*

To prove this we start with the following.

**Proposition 3.2.** *Given $T\in H^{2,2}(X,\mathbb{Z})$, suppose that $\langle h^2,T\rangle\subset H^4(X,\mathbb{Z})$ has rank 2 and is saturated. Let $d=d(T)$ denote its discriminant, and define*

$$M_T:=\left\{\kappa\in K_{\mathrm{num}}(\mathcal{A}_X):c_2(\kappa)\in\langle h^2,T\rangle\right\}.$$

*This has rank 3 by Proposition 2.4. Then the following are equivalent:*

1. *$M_T$ contains a copy of $U$,*
2. *$M_T$ is isomorphic to $U\oplus(-d)=\begin{pmatrix}0&1&\\1&0&\\&&-d\end{pmatrix}$,*
3. *$d$ satisfies (∗∗).*

*Proof.* The implication (2) ⇒ (1) is clear. For (1) ⇒ (2), if $U\subset M_T$ then $M_T$ splits as $U\oplus U^{\perp}$, since $U$ is unimodular. By Proposition 2.5 we have $\operatorname{disc}(M_T)=d$, so $U^{\perp}$ is a rank-1 lattice of discriminant $-d$.

For (2) ⇔ (3) we use Nikulin's theory of discriminant quadratic forms[11]. By [38, Cor. 1.13.4], if $L$ is any even lattice of signature $(r,s)$ then $U\oplus L$ is the *unique* even lattice of signature $(r+1,s+1)$ and discriminant quadratic form $q_{U\oplus L}=q_U\oplus q_L=q_L$. By (∗), $d$ is even and positive, so $U\oplus(-d)$ is an even lattice of signature $(1,2)$. We will show that $M_T$ is also an even lattice of signature $(1,2)$, and that $q_{M_T}\cong q_{(-d)}$ if and only if $d$ satisfies (∗∗).

**Claim:** *$M_T$ is an even lattice of signature $(1,2)$.*

Indeed, $K_{\mathrm{num}}(\mathcal{A}_X)\cong U^{\oplus 4}\oplus E_8{}^{\oplus 2}$ is even, so its sublattice $M_T$ is too. Since $T$ has type $(2,2)$, the Hodge-Riemann bilinear relations imply that $\langle h^2,T\rangle$ is positive definite. Therefore Proposition 2.5 implies that $M_T$ has signature $(1,2)$.

[11]Given an *even* lattice $L$ with its pairing $L\to L^*$, the discriminant form $q_L$ is the finite abelian group $L/L^*$ of order $|\operatorname{disc}(L)|$ together with the induced quadratic form. Here the $2\mathbb{Z}$-valued quadratic form on $L$ induces a $\mathbb{Q}$-valued form on $L^*\supset L$ descending to a $\mathbb{Q}/2\mathbb{Z}$-valued form on $L/L^*$.

**Claim:** $q_{M_T} \cong q_{(-d)}$ *if and only if* $d$ *satisfies* $(**)$.

We have $q_{M_T} \cong q_{-M_T^\perp} \cong q_{-\{h^2,T\}^\perp}$, where the orthogonals are taken in $K_{\mathrm{top}}(\mathcal{A}_X)$ and $H^4(X,\mathbb{Z})$ respectively; the first isomorphism is [38, Cor. 1.6.2] and the second is Proposition 2.3.

But Hassett [19, Prop. 5.1.4] has already proved that $d$ satisfies $(**)$ if and only if $q_{-\{h^2,T\}^\perp} \cong q_{\Lambda_d^0}$, where $\Lambda_d^0 = (-d) \oplus U^{\oplus 2} \oplus (-E_8)^{\oplus 2}$ is the primitive cohomology of a polarised K3 surface of degree $d$. Since $q_{\Lambda_d^0} = q_{(-d)} \oplus q_U^{\oplus 2} \oplus q_{E_8}^{\oplus 2} = q_{(-d)}$ this finishes the proof. □

Thus if $X \in \mathcal{C}_d$ for some $d$ satisfying $(**)$ then $K_{\mathrm{num}}(\mathcal{A}_X)$ contains a copy of $U$. To complete the proof of Theorem 3.1 we suppose, conversely, that $K_{\mathrm{num}}(\mathcal{A}_X)$ contains classes $\kappa_1, \kappa_2$ such that

$$\chi(\kappa_1,\kappa_1) = \chi(\kappa_2,\kappa_2) = 0, \qquad \chi(\kappa_1,\kappa_2) = 1. \tag{3.1}$$

Consider the sublattice $M := \langle h^2, c_2(\kappa_1), c_2(\kappa_2)\rangle \subset H^4(X,\mathbb{Z})$.

If $M$ has rank 1 then by Proposition 2.5, $\kappa_1$ and $\kappa_2$ lie in the negative-definite sublattice $\langle \lambda_1, \lambda_2\rangle$ of $K_{\mathrm{top}}(\mathcal{A}_X)$, which contradicts (3.1).

If $M$ has rank 2 then write its saturation as $\langle h^2, T\rangle$. By Proposition 3.2, $d(T)$ satisfies $(**)$.

Finally, if $M$ has rank 3 then our argument is rather more involved: we will show there exist $x, y \in \mathbb{Z}$ such that[12] $d(xc_2(\kappa_1) + yc_2(\kappa_2))$ satisfies $(**)$.

By Proposition 2.5,

$$d(xc_2(\kappa_1) + yc_2(\kappa_2)) = \mathrm{disc}\langle \lambda_1, \lambda_2, x\kappa_1 + y\kappa_2\rangle.$$

Writing the Euler pairing on $\langle \lambda_1, \lambda_2, \kappa_1, \kappa_2\rangle$ as

$$M := \begin{pmatrix} -2 & 1 & k & m \\ 1 & -2 & l & n \\ k & l & 0 & 1 \\ m & n & 1 & 0 \end{pmatrix}, \tag{3.2}$$

we have

$$d(xc_2(\kappa_1) + yc_2(\kappa_2)) = Ax^2 + Bxy + Cy^2, \tag{3.3}$$

where the coefficients $A$, $B$, and $C$ are the following minors of $M$:

$$A = \begin{vmatrix} -2 & 1 & k \\ 1 & -2 & l \\ k & l & 0 \end{vmatrix}, \qquad B = 2\begin{vmatrix} -2 & 1 & m \\ 1 & -2 & n \\ k & l & 1 \end{vmatrix}, \qquad C = \begin{vmatrix} -2 & 1 & m \\ 1 & -2 & n \\ m & n & 0 \end{vmatrix}.$$

[12] Here we apply our notation $d(T) = \mathrm{disc}\langle h^2, T\rangle = 3T^2 - (h^2.T)^2$ to $T = xc_2(\kappa_1) + yc_2(\kappa_2) \in H^4(X,\mathbb{Z})$. We may have to change this $T$ to ensure that $\langle h^2, T\rangle$ is saturated, with the result that $d(T)$ divides $d(xc_2(\kappa_1) + yc_2(\kappa_2))$. So it is indeed enough to show that the latter satisfies $(**)$; this implies that $d(T)$ does too, and $X \in \mathcal{C}_d$.

We compute

$$\begin{aligned} B^2 - 4AC &= -12 \det M \\ &= -12 \operatorname{disc}\langle \lambda_1, \lambda_2, \kappa_1, \kappa_2 \rangle \\ &= -12 \operatorname{disc}\langle h^2, c_2(\kappa_1), c_2(\kappa_2) \rangle. \end{aligned}$$

Recalling that $c_2(\kappa_1)$ and $c_2(\kappa_1)$ lie in $H^{2,2}(X)$, this is negative by the Hodge-Riemann bilinear relations. Therefore the quadratic form (3.3) is positive definite.

**Lemma 3.3.** *The highest common factor* $h := \operatorname{hcf}(A, B, C)$ *of the coefficients* $A, B, C$ *is even and satisfies* ($**$).

*Proof.* Since the $-A_2$ lattice appears in the top right hand corner of $M$ (3.2), it is convenient to phrase things in terms of the *Eisenstein integers* $\mathbb{Z}[\omega] \subset \mathbb{C}$, where $\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$. Endowing $\mathbb{Z}[\omega]$ with the integral bilinear form $(\alpha, \gamma) \mapsto -2\operatorname{Re}(\alpha\bar{\gamma})$ gives a lattice[13] isomorphic to $-A_2$ with the basis $\{1, \omega\}$ corresponding to $\{\lambda_1, \lambda_2\}$. So we replace the variables $k, l, m, n$ of (3.2) with the Eisenstein integers

$$\alpha := k - l\omega, \qquad \gamma := m - n\omega.$$

In these variables we find that

$$A = 2|\alpha|^2, \qquad B = 4\operatorname{Re}(\alpha\bar{\gamma}) + 6, \qquad C = 2|\gamma|^2.$$

In particular $h = \operatorname{hcf}(A, B, C)$ is even. We will use the following standard facts about the ring $\mathbb{Z}[\omega]$: it is a principal ideal domain in which

(1) $\sqrt{-3} = 1 + 2\omega$ is prime,
(2) every prime $p \in \mathbb{Z}$ with $p \equiv 2 \pmod 3$ is prime in $\mathbb{Z}[\omega]$, and
(3) every $\beta \in \mathbb{Z}[\omega]$ has $|\beta|^2 \equiv 0$ or $1 \pmod 3$.

The last of these is easily checked by listing the elements of $\mathbb{Z}[\omega]$ modulo 3.

Suppose that $p$ is an odd prime with $p \equiv 2 \pmod 3$. Then if $p \mid A$ we have

$$p \mid 2\alpha\bar{\alpha} \Longrightarrow p \mid \alpha \Longrightarrow p \mid 2\operatorname{Re}(\alpha\bar{\gamma}) \Longrightarrow p \nmid B = 4\operatorname{Re}(\alpha\bar{\gamma}) + 6 \Longrightarrow p \nmid h,$$

as required. Similarly we find that $4 \nmid h$.

Suppose that $9 = (\sqrt{-3})^4$ divides both $A = 2\alpha\bar{\alpha}$ and $C = 2\gamma\bar{\gamma}$. Then $\sqrt{-3}$ divides $\alpha$ and $\gamma$ twice, so

$$9 \mid 2\operatorname{Re}(\alpha\bar{\gamma}) \implies 9 \nmid B = 4\operatorname{Re}(\alpha\bar{\gamma}) + 6 \implies 9 \nmid h. \qquad \square$$

We want to show that $Ax^2 + Bxy + Cy^2$ represents[14] a number satisfying ($**$). Let

$$a = A/h, \quad b = B/h, \quad c = C/h.$$

[13] However when we write $|\alpha|^2$ for some $\alpha \in \mathbb{Z}[\omega]$ we mean the usual Euclidean norm rather than this quadratic form.

[14] A quadratic form $Ax^2 + Bxy + Cy^2$ is said to *represent* a number $N$ if there are $x, y \in \mathbb{Z}$ such that $Ax^2 + Bxy + Cy^2 = N$.

Given Lemma 3.3 it is now enough to prove the following.

**Proposition 3.4.** *The positive definite primitive quadratic form* $q(x, y) := ax^2 + bxy + cy^2$ *represents a prime* $p \equiv 1 \pmod 3$.

*Proof.* Define $D = D(q) := b^2 - 4ac$. There are two cases: $3 \mid D$ and $3 \nmid D$.

**Case 1: $3 \mid \mathbf{D}$.** By simply listing the quadratic forms modulo 3 for which $D(q) \equiv 0 \pmod 3$, we find that $q$ represents only

- $0 \pmod 3$ (when $q \equiv 0 \pmod 3$), or
- $0$ and $1 \pmod 3$ ($q \equiv x^2,\ y^2,\ x^2 + xy + y^2,\ x^2 + 2xy + y^2$), or
- $0$ and $2 \pmod 3$ ($q \equiv 2x^2,\ 2y^2,\ 2x^2 + 2xy + 2y^2,\ 2x^2 + xy + 2y^2$),

but not all of $0, 1, 2 \pmod 3$. The first case occurs only when $q$ is not primitive, so we ignore it. In the second case, since any primitive positive definite form represents a prime (in fact infinitely many [14, Thm. 9.12]) it must represent a prime $p \equiv 1 \pmod 3$, as required.

So it is enough to show that $a \equiv 1 \pmod 3$ or $c \equiv 1 \pmod 3$ since both imply that $q$ represents $1 \pmod 3$, putting us in the second case.

Suppose first that $|\alpha|^2, |\gamma|^2$ are both $0 \pmod 3$. Then

$$\sqrt{-3} \mid \alpha, \gamma \implies 3 \mid B \implies 3 \mid h,$$

which by (∗∗) implies that the integer $h/6 \equiv 1 \pmod 3$. Therefore, writing $\alpha = \sqrt{-3}\alpha'$ and $\gamma = \sqrt{-3}\gamma'$, we have

$$|\alpha'|^2 = a(h/6) \equiv a \pmod 3 \quad \text{and} \quad |\gamma'|^2 = c(h/6) \equiv c \pmod 3.$$

So it is enough to show that one of $|\alpha'|^2, |\gamma'|^2$ is $1 \pmod 3$. But if not then by fact (3) above, $\sqrt{-3} \mid \alpha'$ and $\sqrt{-3} \mid \gamma'$, so $3 \mid 2\operatorname{Re}(\alpha'\bar{\gamma}')$, so

$$D \equiv D(h/6)^2 = (2\operatorname{Re}(\alpha'\bar{\gamma}') + 1)^2 - 4|\alpha'|^2|\gamma'|^2 \equiv 1 \pmod 3,$$

contradicting our assumption that $3 \mid D$.

So by fact (3) we are left with the case where at least one of $|\alpha|^2, |\gamma|^2$ is $1 \pmod 3$. Then $h$ is not divisible by 3, so $h/2 \equiv 1 \pmod 3$ by (∗∗). Hence

$$|\alpha|^2 = a(h/2) \equiv a \pmod 3, \quad |\gamma|^2 = c(h/2) \equiv c \pmod 3,$$

and at least one of $a$, $c$ is $1 \pmod 3$.

**Case 2: $3 \nmid \mathbf{D}$.** In this case we adapt the usual proof that $q$ represents infinitely many primes to show that it represents infinitely many congruent to $1 \pmod 3$. We follow [14, Thm. 9.12]. Consider $K := \mathbb{Q}(\sqrt{D})$, the order $\mathcal{O} \subset \mathcal{O}_K$ of discriminant $D$, and its ring class field $L$. Via Artin reciprocity, $q$ corresponds to an element $\sigma_0 \in \operatorname{Gal}(L/K)$. Let $\langle \sigma_0 \rangle$ denote the conjugacy class of its image in $\operatorname{Gal}(L/\mathbb{Q})$. To show that $q$ represents infinitely many primes it is enough to show that the Dirichlet density of

$$\left\{ p \text{ prime} : p \text{ is unramified in } L, \left( \frac{L/\mathbb{Q}}{p} \right) = \langle \sigma_0 \rangle \right\} \tag{3.4}$$

is positive, which is true by the Čebotarev density theorem. To show that $q$ represents infinitely many primes that are 1 (mod 3), we adapt this as follows.

First we claim that 3 is unramified in $L$. Certainly 3 is unramified in $K$, since $3 \nmid D$. Either $3\mathcal{O}_K$ is prime in $\mathcal{O}_K$, or $3\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ for some prime $\mathfrak{p}$ of $\mathcal{O}_K$. In the first case it is enough to show that $3\mathcal{O}_K$ is unramified in $L$, and in the second that $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ are unramified in $L$.

By [14, p. 180], if a prime of $\mathcal{O}_K$ is ramified in $L$ then it divides $f\mathcal{O}_K$, where $f = [\mathcal{O}_K : \mathcal{O}]$ is the conductor of $\mathcal{O}$. Therefore its norm divides $N(f\mathcal{O}_K) = f^2$, hence divides $f^2 d_K = D$, where $d_K$ is the discriminant of $K$. In the two cases above we have $N(3\mathcal{O}_K) = 9$ and $N(\mathfrak{p}) = N(\bar{\mathfrak{p}}) = 3$, neither of which divides $D$.

Since 3 is unramified in $L$ we have $\sqrt{-3} \notin L$, so let $L' = L(\sqrt{-3})$. Then

$$\mathrm{Gal}(L'/\mathbb{Q}) \cong \mathrm{Gal}(L/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q}) = \mathrm{Gal}(L/\mathbb{Q}) \times \{\pm 1\}.$$

Saying that $p \equiv 1 \pmod 3$ is equivalent to saying that the Artin symbol $\left(\frac{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}{p}\right) = 1$. Thus we replace (3.4) with

$$\left\{ p \text{ prime} : p \text{ is unramified in } L', \left(\frac{L'/\mathbb{Q}}{p}\right) = \langle\sigma_0\rangle \times 1 \right\}$$

whose Dirichlet density is once again positive by the Čebotarev density theorem. $\square$

## 4. Non-emptiness

In this Section we will prove that each nonempty Noether-Lefschetz divisor $\mathcal{C}_d$ intersects $\mathcal{C}_8$, and further that the intersection contains cubics $X$ with $\mathcal{A}_X$ geometric.

**Theorem 4.1.** *Suppose that $d$ satisfies ($*$), that is, $d > 6$ and $d \equiv 0$ or $2 \pmod 6$. Then there is an $X \in \mathcal{C}_d \cap \mathcal{C}_8$ such that $\mathcal{A}_X$ is geometric*[15].

A cubic $X \in \mathcal{C}_8$ contains a plane $P$ [43, §3]. The linear system of hyperplanes containing $P$ defines a map $\mathrm{Bl}_P(X) \to \mathbb{P}^2$ whose fibres are quadric surfaces $Q$ (a point of $\mathbb{P}^2$ – the dual of the linear system – corresponds to two hyperplanes whose intersection is the reducible cubic surface $Q \cup P$). We also let $Q = h^2 - P$ denote the class in $X$ (rather than $\mathrm{Bl}_P(X)$) of a smooth fibre. We have $P^2 = 3$ and $Q^2$=4.

The fibres degenerate over a sextic curve $C \subset \mathbb{P}^2$; if it is smooth then the double cover of $\mathbb{P}^2$ branched over $C$ is a K3 surface $S$, and there is a natural Brauer class $\alpha \in H^2_{\mathrm{an}}(S, \mathcal{O}_S^*)_{\mathrm{tors}}$. Kuznetsov [27, §4] has shown that $\mathcal{A}_X \cong D(S, \alpha)$, where the latter is the derived category of $\alpha$-twisted sheaves.

[15] Thus by the easy direction of Theorem 1.1, which was proved in Section 3, there is another class $T' \in H^{2,2}(X, \mathbb{Z})$ such that $d' = d(T')$ satisfies ($**$). In fact we can produce $T' \in \langle Q, T\rangle$ more directly by using (4.4) and arguing as in "Case 1" of Proposition 3.4.

To prove the Theorem it is enough to find such an $X \in \mathcal{C}_8$ with a class $T \in H^{2,2}(X,\mathbb{Z})$ such that $d(T) = d$ and $T.Q = 1$: by [27, Prop. 4.7] this implies that $\alpha = 0$ and so $\mathcal{A}_X$ is geometric.

So we fix $L := H^4(Y,\mathbb{Z})$ for some cubic fourfold $Y$ containing a plane, with corresponding classes $h^2, P \in L$, and $Q = h^2 - P$. We will first produce a suitable $T \in L$, and then find another cubic $X$ with these classes in $H^{2,2}(X,\mathbb{Z})$ using Laza and Looijenga's description of the image of the period map. We begin with the following[16].

**Proposition 4.2.** *For any $n > 0$ with $n \equiv 5 \pmod 8$, there is a $T \in L$ such that $T.Q = 1$ and $\langle h^2, Q, T\rangle \subset L$ is saturated of discriminant $n$ with intersection pairing*

$$\begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 2k \end{pmatrix} \qquad \text{when } n = 16k-3. \tag{4.1}$$

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 2k+1 \end{pmatrix} \qquad \text{when } n = 16k+5, \text{ or} \tag{4.2}$$

*In particular, $d(T) = 6k+2$ in (4.2) and $d(T) = 6k$ in (4.1).*

*Proof.* Hassett [18, §4] already gives us $T$ with $T.Q$ odd and $\langle h^2, Q, T\rangle \subset L$ saturated of discriminant $n$. The intersection pairing has the form

$$\begin{pmatrix} 3 & 2 & * \\ 2 & 4 & 2a+1 \\ * & 2a+1 & * \end{pmatrix}.$$

Replacing $T$ with $T - ah^2$, this becomes

$$\begin{pmatrix} 3 & 2 & 4b+c \\ 2 & 4 & 1 \\ 4b+c & 1 & * \end{pmatrix}$$

for some $0 \le c \le 3$. Replacing $T$ with $T - 2bh^2 + bQ$ gives

$$\begin{pmatrix} 3 & 2 & c \\ 2 & 4 & 1 \\ c & 1 & * \end{pmatrix}. \tag{4.3}$$

Since changing $T$ to $h^2 - T$ replaces $c$ with $3 - c$ in (4.3), we may assume without loss of generality that $c$ is either 0 or 1.

Now $d(T) = 3T^2 - c^2$ is necessarily even. (In fact $3d(T) = T'^2$ where $T' := 3T - (h^2.T)h^2$ lies in $\{h^2\}^\perp$, which is an *even* lattice [19, Prop. 2.1.2].) Thus if $c = 0$ then $T^2$ is even, so setting $2k = T^2$ we get (4.1). If $c = 1$ then $T^2$ is odd, so setting $2k+1 = T^2$ we get (4.2). □

[16] Since $d(T) > 6$ (*) this result confirms Hassett's expectation [18, Rem. 4.3.2] that $n = 5, 13$ do not occur in the locus of smooth cubics, but, by the proof of Theorem 4.1, all larger $n \equiv 5 \pmod 8$ do.

*Proof of Theorem 4.1.* Since the $d = 8$ case is trivial we fix $d > 8$ with $d \equiv 0$ or $2 \pmod 6$. We apply Proposition 4.2 with $n = 16k - 3$ (if $d = 6k$ with $k \geq 2$) or $n = 16k + 5$ (if $d = 6k + 2$ with $k \geq 2$).

This yields $T \in L$ with discriminant $d$ and intersection form (4.1) or (4.2) on $\langle h^2, Q, T\rangle$. Thus for any $T' = xh^2 + yQ + zT$ we have

$$d(T') = 8y^2 + 6yz + 6kz^2 \quad \text{or} \quad d(T') = 8y^2 + 2yz + (6k+2)z^2 \tag{4.4}$$

respectively. By the theory of reduced quadratic forms [5, §5.2], the smallest nonzero value taken by these forms is 8, at only $y = \pm 1, z = 0$. In particular,

$$\nexists\ T' \in \langle h^2, Q, T\rangle \text{ with } d(T') = 2 \text{ or } 6. \tag{4.5}$$

Next we produce a period point $\sigma \in L \otimes \mathbb{C}$ with

$$\sigma^2 = 0,\ \sigma\bar{\sigma} < 0 \ \text{ and } \ L \cap \sigma^\perp = \langle h^2, Q, T\rangle.$$

Recall that the signature of $L$ is $(21, 2)$. Choose a 21-dimensional positive definite real subspace $V \subset L \otimes \mathbb{R}$ such that $V \cap L = \langle h^2, Q, T\rangle$. Then the pairing is negative definite on $V^\perp \subset L \otimes \mathbb{R}$, so any nonzero $\sigma \in V^\perp \otimes \mathbb{C} \subset L \otimes \mathbb{C}$ has $\sigma\bar{\sigma} < 0$. Moreover, the line $\mathbb{P}(V^\perp \otimes \mathbb{C}) \subset \mathbb{P}(L \otimes \mathbb{C})$ necessarily meets the quadric $\{\sigma^2 = 0\}$.

Then by Laza and Looijenga's description of the image of the period map [29, Thm. 1.1], [32, Thm. 3.1], the condition (4.5) guarantees the existence of a cubic $X$ and an isometry $L \cong H^4(X, \mathbb{Z})$ preserving $h^2$ and taking $\sigma$ to a generator of $H^{3,1}(X)$. In particular, $\langle h^2, Q, T\rangle = H^{2,2}(X, \mathbb{Z})$ by the construction of $V = \langle \sigma, \bar{\sigma}\rangle^\perp$, and by [43, §3] there is a plane $P \subset X$ with cohomology class $P = h^2 - Q$.

Finally we claim that the discriminant sextic of $\mathrm{Bl}_P(X) \to \mathbb{P}^2$ is smooth. Voisin [43, §1] observes that it is enough to show that there is no other plane $P' \subset X$ meeting $P$, and that distinct planes have independent homology classes. Such a $P' \in \langle h^2, Q, T\rangle$ would have discriminant $d(P') = 8$, which by our discussion of reduced quadratic forms (4.4) implies that $P' = xh^2 \pm Q$. From $h^2.P' = 1$ we deduce that $P' = h^2 - Q = P$, as desired. □

## 5. Setup for deformation theory

5.1. **Modification of the equivalence.** Having found $X \in \mathcal{C}_d \cap \mathcal{C}_8$ with $\mathcal{A}_X$ geometric, as described in Section 1.2 we now modify the equivalence $\Phi\colon D(S) \xrightarrow{\sim} \mathcal{A}_X$ to the "right one" – the one which should deform as $X$ deforms in $\mathcal{C}_d$ – when $d$ satisfies (∗∗).

A necessary condition is that $c_2(\Phi(\mathcal{O}_{\text{point}}))$, $c_2(\Phi(\mathcal{I}_{\text{point}}))$ should lie in $\langle h^2, T\rangle$ so that they remain algebraic as we deform $X$ in $\mathcal{C}_d$. We achieve this condition in Proposition 5.2 below. Ultimately we shall find that it is also sufficient to allow us to deform $\Phi$.

To modify $\Phi$ we will apply Mukai's results on moduli of sheaves on K3 surfaces. These only apply in rank $\geq 1$, so we need the following trick of Orlov.

**Lemma 5.1.** [39, §3.7] *Suppose that $\mathcal{A}_X$ is geometric, and $\kappa_1 \in K_{\mathrm{num}}(\mathcal{A}_X)$ is any point-like class (i.e. $\chi(\kappa_1, \kappa_1) = 0$).*

*Then there is a K3 surface $S$ and equivalence $\Phi : D(S) \xrightarrow{\sim} \mathcal{A}_X$ with*

$$\operatorname{rank}(\Psi^K(\kappa_1)) \geq 1,$$

*where $\Psi\colon D(X) \to D(S)$ is the right adjoint of $\Phi$.*

*Proof.* Pick any equivalence $\Phi : D(S) \xrightarrow{\sim} \mathcal{A}_X$ and write

$$v(\Psi^K(\kappa_1)) = (r, \ell_1, s) \in H^*(S, \mathbb{Z}).$$

If $r < 0$, replace $\Phi$ with $\Phi[1]$. If $r = 0$ and $s \neq 0$, precompose $\Phi$ with Mukai's reflection functor (i.e. the spherical twist around $\mathcal{O}_S$), so $v(\Psi^K(\kappa_1))$ becomes $(s, \ell_1, 0)$ and we are reduced to the previous case. If $r = s = 0$ then

$$\ell_1^2 = -\chi(\Psi^K(\kappa_1), \Psi^K(\kappa_1)) = 0,$$

so for any ample $\ell \in \operatorname{Pic}(S)$ we have $\ell_1.\ell \neq 0$ by the Hodge index theorem. Replace $\Phi$ with $\Phi \circ (\,\cdot\, \otimes \ell^\vee)$, so $v(\Psi^K(\kappa_1))$ becomes

$$(0, \ell_1, 0) \cup (1, \ell, \tfrac{1}{2}\ell^2) = (0, \ell_1, \ell_1.\ell)$$

and we are again reduced to the previous case. $\square$

**Proposition 5.2.** *Fix a cubic fourfold $X$ and a saturated sublattice $\langle h^2, T\rangle \subset H^{2,2}(X, \mathbb{Z})$ such that $d = d(T)$ satisfies (**).*

*If $\mathcal{A}_X$ is geometric then there is a polarised K3 surface $(S, \ell)$ of degree $d$ and an equivalence $\Phi\colon D(S) \xrightarrow{\sim} \mathcal{A}_X$ such that*

$$c_2(\Phi(\mathcal{O}_{point})),\ c_2(\Phi(\mathcal{I}_{point})),\ c_2(\Phi(\ell)) \ \in\ \langle h^2, T\rangle.$$

*In particular, $\Phi$ induces an isometry*

$$H^2(S, \mathbb{Z}) \supset\ \ell^\perp \ \xrightarrow[\sim]{\Phi^{H^*}}\ \{h^2, T\}^\perp\ \subset H^4(X, \mathbb{Z}). \tag{5.1}$$

*Proof.* By Proposition 3.2 we can choose a basis $\kappa_1, \kappa_2, \kappa_3$ for the lattice

$$\{\kappa \in K_{\mathrm{num}}(\mathcal{A}_X) : c_2(\kappa) \in \langle h^2, T\rangle\}$$

in which the Euler pairing is

$$\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -d \end{pmatrix}.$$

Let $\Phi' : S' \xrightarrow{\sim} \mathcal{A}_X$ and $\Psi'$ be as in Lemma 5.1, so $\operatorname{rank}(\Psi'^K(\kappa_1)) \geq 1$. Since $\chi(\Psi'^K(\kappa_1), \Psi'^K(\kappa_2)) = 1$, the standard theory of moduli spaces of sheaves on projective varieties, nicely summarised in [23, §10.3], gives an ample line bundle $\ell'$ on $S'$ such that the moduli space $S$ of $\ell'$-semi-stable sheaves on $S'$ belonging to the class $\Psi'^K(\kappa_1)$ is projective, fine, and contains only strictly stable sheaves.

As $\chi(\Psi'^K(\kappa_1), \Psi'^K(\kappa_1)) = 0$ and $\operatorname{rank}(\Psi'^K(\kappa_1)) \geq 1$, Mukai proves this moduli space $S$ is non-empty [37, Thm. 5.1] and a K3 surface [37, Thm. 1.4]. The universal sheaf induces an equivalence $D(S) \xrightarrow{\sim} D(S')$ [37, Prop. 4.10];

compose this with $\Phi'$ to give a preliminary $\Phi\colon D(S) \xrightarrow{\sim} \mathcal{A}_X$ satisfying $\Phi^K[\mathcal{O}_{\mathrm{point}}] = \kappa_1$.

Let $\Psi$ be the right adjoint of $\Phi$, so that $\Psi^K(\kappa_1) = [\mathcal{O}_{\mathrm{point}}]$. Writing

$$v(\Psi^K(\kappa_2)) = (r, \ell_2, s),$$

then

$$r = \chi(\Psi^K(\kappa_1), \Psi^K(\kappa_2)) = 1 \quad \text{and} \quad 2s - \ell_2^2 = \chi(\Psi^K(\kappa_2), \Psi^K(\kappa_2)) = 0,$$

i.e. $v(\Psi^K(\kappa_2)) = (1, \ell_2, \ell_2^2/2)$. So if we replace $\Phi$ with $\Phi \circ (\,\cdot\, \otimes \ell_2)$ we find that $\Phi^K[\mathcal{O}_{\mathrm{point}}] = \kappa_1$ and $\Phi^K[\mathcal{I}_{\mathrm{point}}] = \kappa_2$.

Since $\kappa_3$ is orthogonal to $\kappa_1, \kappa_2$ we see that $v(\Psi^K(\kappa_3))$ is orthogonal to

$$v(\mathcal{O}_{\mathrm{point}}) = (0,0,1) \qquad \text{and} \qquad v(\mathcal{I}_{\mathrm{point}}) = (1,0,0).$$

Thus $v(\Psi^K(\kappa_3)) = (0, \ell, 0)$ for some $\ell \in H^{1,1}(S,\mathbb{Z})$. We get a commutative diagram

$$\begin{tikzcd}
K_{\mathrm{top}}(\mathcal{A}_X) \supset \{\kappa_1,\kappa_2,\kappa_3\}^\perp \arrow[r, "\Psi^K", "\cong"'] \arrow[d, "v"', "\cong"] & \{[\mathcal{O}_{\mathrm{point}}], [\mathcal{I}_{\mathrm{point}}], \ell\}^\perp \subset K_{\mathrm{top}}(S) \arrow[d, "\cong"', "v"] \\
H^4(X,\mathbb{Z}) \supset \{h^2, T\}^\perp \arrow[r, "\Psi^{H^*}"] & \{(0,0,1),(1,0,0),(0,\ell,0)\}^\perp \subset H^*(S,\mathbb{Z}),
\end{tikzcd}$$

where the bottom right hand term is $\ell^\perp \subset H^2(S,\mathbb{Z})$, and the left hand isometry is Proposition 2.3. It follows that the bottom arrow is an isometry. Taking its inverse gives (5.1).

Finally,

$$-\ell^2 = \chi(\Psi^K(\kappa_3), \Psi^K(\kappa_3))) = \chi(\kappa_3, \kappa_3) = -d.$$

Therefore to show that $\ell$ is ample it is enough to prove there is no class $\delta \in H^{1,1}(S,\mathbb{Z}) \cap \ell^\perp$ with $\delta^2 = -2$. Equivalently, there is no integral class $\delta \in H^{2,2}(X,\mathbb{Z}) \cap \{h^2, T\}^\perp$ with $\delta^2 = 2$: but this is [43, §4, Prop. 1]. (Or observe that $d(\delta)$ would be 6, which contradicts ($*$).) $\square$

5.2. **Construction of the families.** Now fix a $d$ satisfying ($**$) and an $X_0 \in \mathcal{C}_d$ such that $\mathcal{A}_{X_0}$ is geometric. Proposition 5.2 gives us a polarised K3 surface $(S_0, \ell_0)$ and an equivalence $\Phi_0 : D(S_0) \xrightarrow{\sim} \mathcal{A}_{X_0}$ whose right adjoint $\Psi_0$ induces an anti-isometry[17]

$$\Psi_0^{H^*}\colon \{h^2, T\}^\perp \longrightarrow \{\ell_0\}^\perp. \tag{5.2}$$

As described in Section 1.2, we now want to deform $X_0$ inside $\mathcal{C}_d$ and take a corresponding deformation of $S_0$. Since $\mathcal{C}_d$ is not a fine moduli space (a technical inconvenience in Section 7.2) we pass to a finite cover.

[17]In this section we use the intersection pairing on $H^2(S,\mathbb{Z})$. This differs from the Mukai pairing by a sign, making the isometry of Proposition 5.2 into an anti-isometry.

**Theorem 5.3.** *There is a smooth quasi-projective variety $\mathcal{C}_d^{\mathrm{lev}}$ with a finite cover $\pi : \mathcal{C}_d^{\mathrm{lev}} \to \mathcal{C}_d$ and a universal family of cubics*

$$p_X : \mathcal{X} \to \mathcal{C}_d^{\mathrm{lev}}.$$

*That is, the fibre $X_t$ over any $t \in \mathcal{C}_d^{\mathrm{lev}}$ is isomorphic to the cubic represented by the point $\pi(t) \in \mathcal{C}_d$.*

*Moreover, choosing a lift $0 \in \mathcal{C}_d^{\mathrm{lev}}$ of the point $X_0 \in \mathcal{C}_d$, there is a family $p_S : (\mathcal{S}, \ell) \to \mathcal{C}_d^{\mathrm{lev}}$ of smooth polarised K3s, extending $(S_0, \ell_0)$ over $0$, such that the embedding $\Phi_0^{H^*} : H^*(S_0, \mathbb{Q}) \hookrightarrow H^*(X_0, \mathbb{Q})$ extends to an embedding of local systems*

$$Rp_{S*}\mathbb{Q} \hookrightarrow Rp_{X*}\mathbb{Q}$$

*whose complexification takes the period point $H^{2,0}(S_t) \subset H^*(S_t, \mathbb{C})$ to the period point $H^{3,1}(X_t) \subset H^*(X_t, \mathbb{C})$ over any point $t \in \mathcal{C}_d^{\mathrm{lev}}$.*

*Proof.* Consider the lattices

$$L_0 := \{h^2\}^\perp \subset H^4(X_0, \mathbb{Z}) =: L$$

and

$$\Lambda_0 := \{\ell_0\}^\perp \subset H^2(S_0, \mathbb{Z}) =: \Lambda$$

and the associated local period domains

$$D := \{\sigma \in \mathbb{P}(L_0 \otimes \mathbb{C}) : \sigma^2 = 0,\, \sigma\bar{\sigma} < 0\},$$
$$\Delta := \{\sigma \in \mathbb{P}(\Lambda_0 \otimes \mathbb{C}) : \sigma^2 = 0,\, \sigma\bar{\sigma} > 0\}.$$

Let $\mathcal{C}^{\mathrm{mar}}$ be the moduli space of marked cubic fourfolds, that is, cubics $X$ with an isometry $H^4(X, \mathbb{Z}) \cong L$ preserving $h^2$. The period map embeds $\mathcal{C}^{\mathrm{mar}}$ as an open subset of $D$ [43].

Similarly, let $\mathcal{K}_d^{\mathrm{mar}}$ be the moduli space of marked polarised K3 surfaces of degree $d$. The period map embeds $\mathcal{K}_d^{\mathrm{mar}}$ as an open subset of $\Delta$. Its image is the complement of the hyperplane sections $\delta^\perp$, where $\delta \in \Delta$ is any class with $\delta^2 = -2$.

We identify both moduli spaces $\mathcal{C}^{\mathrm{mar}}$, $\mathcal{K}_d^{\mathrm{mar}}$ with their images in the period domains. We claim they both carry universal families. For $\mathcal{K}_d^{\mathrm{mar}}$ this is well-known, and the same argument of [7, Exp. XIII] or [22, §6.3.3] works for $\mathcal{C}^{\mathrm{mar}}$: we glue universal local deformations as follows.

Let $X \in \mathcal{C}^{\mathrm{mar}}$ be a marked cubic and consider the versal deformation $\mathcal{X} \to \mathrm{Def(X)}$. Since $H^0(T_X) = 0$, Def(X) is universal; since $H^2(T_X) = 0$, it is smooth. By the local Torelli theorem the period map embeds Def(X) as an open subset of $D$. Any small deformation of $X$ is another smooth cubic 4-fold, so $h^1(T_{X_t})$ is constant for all $t \in \mathrm{Def(X)}$, so Def(X) is also universal for each of its fibres. Finally and most importantly, $\mathrm{Aut}(X)$ preserves $h^2 \in H^4(X, \mathbb{Z})$ and acts faithfully on $H^4_{\mathrm{prim}}(X, \mathbb{Z})$ [43, p. 600]. In particular, *the only automorphism of $X_t$ which preserves the marking is the identity*, so the local families glue uniquely to give a global universal family $\mathcal{X}^{\mathrm{mar}} \to \mathcal{C}^{\mathrm{mar}}$.

Next define[18] $\mathcal{C}_d^{\rm mar} := \mathcal{C}^{\rm mar} \cap T^{\perp}$, which parametrises marked cubics with a distinguished integral $(2,2)$-class of discriminant $d$. Then (5.2) gives an isomorphism

$$\Psi_0^{H^*}\colon D \cap T^{\perp} \xrightarrow{\sim} \Delta. \tag{5.3}$$

By [43, §4, Prop. 1], a smooth cubic has no integral $(2,2)$ class $\delta$ with $h^2.\delta = 0$ and $\delta^2 = 2$, so (5.3) gives an inclusion

$$\Psi_0^{H^*}(\mathcal{C}_d^{\rm mar}) \subset \mathcal{K}_d^{\rm mar}. \tag{5.4}$$

Thus in addition to the family $\mathcal{X}_d^{\rm mar} := \mathcal{X}^{\rm mar}|_{\mathcal{C}_d^{\rm mar}}$ of cubics we get a family $\mathcal{S}^{\rm mar} \to \mathcal{C}_d^{\rm mar}$ of K3s such that the period points coincide under $\Psi_0^{H^*}$.

Consider the subgroup $G$ of $O(L)$ fixing $h^2$ and $T$. By [38, Cor. 1.5.2] this can be identified with the subgroup of $O(\{h^2, T\}^{\perp})$ acting trivially on the discriminant group of $\{h^2, T\}^{\perp}$. Via $\Psi_0^{H^*}$ this is identified with the subgroup of $O(\Lambda_0)$ acting trivially on the discriminant group of $\Lambda_0$, and hence with the subgroup of $O(\Lambda)$ fixing $\ell_0$. Observe that the embedding (5.4) is $G$-equivariant.

The action of $G$ on $\mathcal{C}_d^{\rm mar}$ lifts to an action on the families $\mathcal{X}_d^{\rm mar}$ and $\mathcal{S}^{\rm mar}$ by the strong forms of the global Torelli theorems. These state that every Hodge isometry $H^4(X_1, \mathbb{Z}) \cong H^4(X_2, \mathbb{Z})$ preserving $h^2$ is induced by a unique isomorphism $X_1 \cong X_2$, and every Hodge isometry $H^2(S_1, \mathbb{Z}) \cong H^2(S_2, \mathbb{Z})$ preserving a polarisation is induced by a unique isomorphism $(S_1, \ell_1) \cong (S_2, \ell_2)$.

Similarly, for any cubic $X$, the stabiliser in $G$ of the period point is $\mathrm{Aut}(X)$. This is finite by [36], so the action of $G$ on $\mathcal{C}_d^{\rm mar}$ has finite stabilisers. Therefore we fix $N \in \mathbb{Z}$ large enough that the group

$$G_N = \{g \in G : g \equiv 1 \pmod N\}$$

is torsion-free, and set $\mathcal{C}_d^{\rm lev} = \mathcal{C}_d^{\rm mar}/G_N$. Now $G_N$ acts freely on $\mathcal{C}_d^{\rm mar}$, so $\mathcal{C}_d^{\rm lev}$ is smooth, and the families descend to $\mathcal{C}_d^{\rm lev}$: take $\mathcal{X} = \mathcal{X}_d^{\rm mar}/G_N$ and $\mathcal{S} = \mathcal{S}^{\rm mar}/G_N$. Moreover, since $G_N$ is a finite-index subgroup of $O(\Lambda_0)$, $\mathcal{C}_d^{\rm lev}$ is quasi-projective by work of Baily and Borel. □

## 6. (Hochschild) (co)homology

In this section and the next we follow the wonderful papers [42, 24][19], but with a number of simplifications and one generalisation (to the fully faithful case: Toda and Huybrechts-Macrì-Stellari deal with equivalences only). We also use $T^1$-lifting methods to simplify the higher order deformation theory. We therefore give a self contained (and hopefully easier-to-follow) account.

[18] Note that this is not what Hassett calls $\mathcal{C}_d^{\rm mar}$ in [19]; his is a quotient of ours.

[19] There is also the related work [2], but this uses a very different description of deformations using *-quantisations.

**Notation: families.** Although we suppress mention of it throughout, we work relative to a complex affine base $B$ in this section and the next. So for instance a "smooth projective variety" means a smooth projective family over $B$. All products, diagonals, etc. are taken relative to $B$; thus $\Delta_S$ denotes the diagonal in $S \times_B S$, etc. Usually $B$ can be taken to be $\operatorname{Spec}\mathbb{C}$, but in Section 7.2 we will apply our results to the case $B = \operatorname{Spec} A$ for an Artinian local $\mathbb{C}$-algebra $A$. The notation is unaffected; we simply replace our scalars $\mathbb{C}$ by $A = \mathcal{O}_B$.

Since we sometimes work with smooth families over *singular* bases $B$, it is important that by $D(Y)$ we will always mean the bounded derived category of *perfect* complexes of coherent sheaves on $Y$. (Note that the structure sheaf of the diagonal $\mathcal{O}_{\Delta_Y}$ is indeed perfect in $D(Y \times_B Y)$.) For $B$ Artinian all cohomologies and Exts will therefore be finite dimensional.

6.1. **Hochschild cohomology.** Suppose that $S$ and $X$ are smooth projective varieties and that $P \in D(S \times X)$ is the Fourier-Mukai kernel of a *fully faithful* embedding $D(S) \hookrightarrow D(X)$.

The usual Fourier-Mukai convolution product $*$, corresponding to composition of Fourier-Mukai functors, gives the functors

$$\begin{array}{ccccc} D(S\times S) & \xrightarrow{P*} & D(S\times X) & \xleftarrow{*P} & D(X\times X), \\ \mathcal{O}_{\Delta_S} & \longmapsto & P & \longleftarrow\!\shortmid & \mathcal{O}_{\Delta_X}. \end{array} \tag{6.1}$$

The first induces a map

$$\operatorname{Ext}^*(\mathcal{O}_{\Delta_S}, \mathcal{O}_{\Delta_S}) \xrightarrow{P*} \operatorname{Ext}^*(P,P) \tag{6.2}$$

which is an isomorphism. In fact[20]

$$R\pi_{1*} R\mathcal{H}om_{S\times S}(\mathcal{O}_{\Delta_S}, \mathcal{O}_{\Delta_S}) \xrightarrow{P*} \pi_{1*} R\mathcal{H}om_{S\times X}(P,P)$$

is a quasi-isomorphism because the full and faithful condition ensures that it is a quasi-isomorphism when restricted to any $s \in S$, where it is the map $\operatorname{Ext}^*(\mathcal{O}_s, \mathcal{O}_s) \to \operatorname{Ext}^*(P_s, P_s)$.

The right hand map of (6.1) gives a map $\operatorname{Ext}^*(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) \to \operatorname{Ext}^*(P,P)$. Combining with the inverse of the isomorphism (6.2) gives[21]

$$HH^*(X) := \operatorname{Ext}^*(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) \longrightarrow \operatorname{Ext}^*(\mathcal{O}_{\Delta_S}, \mathcal{O}_{\Delta_S}) =: HH^*(S). \tag{6.3}$$

We will call this map $\Phi_P^{HH^*}$, and understand it via the variation of Hodge structure it induces. To access this we first go via Hochschild homology.

[20] Alternatively use the right adjoint $\Phi_R$ (6.6), which is a left inverse to the fully faithful $\Phi_P$. Thus $R * P \cong \mathcal{O}_{\Delta_X}$ and $R*$ is also right adjoint to $P*$. Therefore (6.2) is the isomorphism $\operatorname{Ext}^*(\mathcal{O}_{\Delta_S}, \mathcal{O}_{\Delta_S}) \cong \operatorname{Ext}^*(\mathcal{O}_{\Delta_S}, R * P) \cong \operatorname{Ext}^*(P,P)$.

[21] Hochschild cohomology is not in general functorial, but here we see that it *is* under full and faithful functors (as well as equivalences). For $HH^2$ we interpret this as saying that a deformation of $D(X)$ induces a deformation of the full subcategory $D(S)$.

6.2. **Hochschild homology.** Let $S$ and $X$ have dimensions $m$ and $n$ respectively. Hochschild homology is defined by $HH_*(S) := \mathrm{Ext}^{m-*}(\Delta_{S*}\omega_S^{-1}, \mathcal{O}_{\Delta_S})$ and *is* functorial. That is, we get a map $\Phi_P^{HH_*}$

$$\begin{array}{ccc} \mathrm{Ext}^{m-*}(\Delta_{S*}\omega_S^{-1}, \mathcal{O}_{\Delta_S}) & \longrightarrow & \mathrm{Ext}^{n-*}(\Delta_{X*}\omega_X^{-1}, \mathcal{O}_{\Delta_X}) \\ \| & & \| \\ HH_*(S) & \xrightarrow{\Phi_P^{HH_*}} & HH_*(X), \end{array} \tag{6.4}$$

by the following construction [15]. We modify diagram (6.1) to

$$\begin{array}{ccccc} D(S\times S) & \xrightarrow{P*} & D(S\times X) & \xrightarrow{*R} & D(X\times X) \\ \mathcal{O}_{\Delta_S} & \longmapsto & P & \longmapsto & P*R \\ \Delta_{S*}\omega_S^{-1} & \longmapsto & P\otimes\omega_S^{-1} & \longmapsto & P*L\otimes\pi_1^*\omega_X^{-1}[m-n]. \end{array} \tag{6.5}$$

Here, if $\tau$ denotes the isomorphism $X\times S\to S\times X$, then

$$R := \tau^*(P^\vee\otimes\omega_S[m]) \in D(X\times S) \tag{6.6}$$

is the kernel for the right adjoint of $\Phi_P$, and

$$L := \tau^*(P^\vee\otimes\omega_X[n]) = R\otimes\omega_S^{-1}\otimes\omega_X[n-m]$$

is the kernel for the left adjoint. From (6.5) we get a map

$$\mathrm{Ext}^{m-*}(\Delta_{S*}\omega_S^{-1}, \mathcal{O}_{\Delta_S}) \to \mathrm{Ext}^{m-*}(P*L\otimes\pi_1^*\omega_X^{-1}[m-n], P*R).$$

Compose with the natural maps of kernels

$$\mathcal{O}_{\Delta_X}\xrightarrow{\eta} P*L, \qquad P*R\xrightarrow{\epsilon}\mathcal{O}_{\Delta_X} \tag{6.7}$$

that induce the unit and counit of the adjunctions. This takes us to the group $\mathrm{Ext}^{n-*}(\Delta_{X*}\omega_X^{-1}, \mathcal{O}_{\Delta_X}) = HH_*(X)$, thus defining the map (6.4).

For any variety $Y$ there is an action of $HH^*(Y)$ on $HH_*(Y)$ given by composition:

$$\mathrm{Ext}^i(\Delta_{Y*}\omega_Y^{-1}, \mathcal{O}_{\Delta_Y})\otimes\mathrm{Ext}^j(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{\Delta_Y}) \longrightarrow \mathrm{Ext}^{i+j}(\Delta_{Y*}\omega_Y^{-1}, \mathcal{O}_{\Delta_Y}). \tag{6.8}$$

This action has a certain compatibility with the maps $\Phi_P^{HH^*}$ (6.3) and $\Phi_P^{HH_*}$ (6.4), as described in the next result.

**Proposition 6.1.** *Fix $P\in D(S\times X)$ and $e_X\in HH^j(X)$. The following diagram commutes:*

$$\begin{array}{ccc} HH_i(S) & \xrightarrow{\Phi_P^{HH_*}} & HH_i(X) \\ {\scriptstyle \Phi_P^{HH^*}(e_X)}\big\downarrow & & \big\downarrow{\scriptstyle e_X} \\ HH_{i-j}(S) & \xrightarrow{\Phi_P^{HH_*}} & HH_{i-j}(X). \end{array} \tag{6.9}$$

*Proof.* Fix any $f_S \in HH_i(S)$ mapping via (6.4) to $f_X \in HH_i(X)$. Consider $f_S$ as a map from the bottom left hand object $\Delta_{S*}\omega_S^{-1}$ in the diagram (6.5) to the central left hand object $\mathcal{O}_{\Delta_S}[m-i]$. Compose with $\Phi_P^{HH^*}(e_X) : \mathcal{O}_{\Delta_S} \to \mathcal{O}_{\Delta_S}$ and follow the composition from left to right through (6.5). We get

$$\begin{tikzcd}
\mathcal{O}_{\Delta_S} & P & P*R \arrow[r, "\epsilon"] & \mathcal{O}_{\Delta_X} \\
\mathcal{O}_{\Delta_S} \arrow[u, "\Phi_P^{HH^*}(e_X)"] \arrow[r, mapsto, "P*"] & P \arrow[u, "e_X*1_P"] \arrow[r, mapsto, "*R"] & P*R \arrow[u, "e_X*1_P*1_R"] \arrow[r, "\epsilon"] & \mathcal{O}_{\Delta_X} \arrow[u, "e_X"'] \\
\Delta_{S*}\omega_S^{-1} \arrow[u, "f_S"] & P \otimes \omega_S^{-1} \arrow[u] & P*L \otimes \pi_1^*\omega_X^{-1}[n-m] \arrow[u] & \\
 & & \Delta_{X*}\omega_X^{-1}[n-m]. \arrow[u, "\eta"] &
\end{tikzcd}$$

Applying $P*$ to $\Phi_P^{HH^*}(e_X)$ (top left) gives $e_X * 1_P$ (top centre) by the definition of $\Phi_P^{HH^*}(e_X)$ via diagram (6.1). This explains the labelling of the arrows. We are required to prove that the full composition up-up-up-right on the right hand side is $e_X \circ f_X$. It is the same as up-up-right-up because the top right square clearly commutes. But up-up-right is $f_X$ by definition. $\square$

6.3. **Hodge cohomology.** We can reinterpret the results of the last two sections via the *modified* Hochschild-Kostant-Rosenberg isomorphisms

$$HH^*(Y) \;\cong \bigoplus_{i+j=*} H^i(\Lambda^j T_Y), \qquad HH_*(Y) \;\cong \bigoplus_{j-i=*} H^i(\Omega_Y^j), \tag{6.10}$$

for any smooth projective variety $Y$. Here we post-compose the standard HKR isomorphism (given by the exponential of the universal Atiyah class [11, 16]) with $\mathrm{td}^{-1/2} \lrcorner(\,\cdot\,)$ acting on $\bigoplus H^i(\Lambda^j T_Y)$ and $\mathrm{td}^{1/2} \wedge(\,\cdot\,)$ acting on $\bigoplus H^i(\Omega_Y^j)$. Whenever we write something like $H^1(T_Y) \subset HH^2(Y)$ we mean via this modified HKR isomorphism[22].

For us, the modified HKR isomorphism has two main advantages over its untwisted cousin. Firstly [13, 12], it intertwines the action (6.8) of $HH^*(Y)$ on $HH_*(Y)$ with the interior multiplication of $H^*(\Lambda^* T_Y)$ on $H^*(\Omega_Y^*)$. Secondly [33, Thm. 1.2], it intertwines the map $\Phi_P^{HH_*}$ (6.4) on Hochschild homology with the usual map $\Phi_P^{H^*}$ (1.2) on cohomology.

From now on we restrict attention to the case of interest: $S$ a K3 surface, $X$ a cubic 4-fold, and $P \in D(S \times X)$ the kernel of a full and faithful embedding $D(S) \to D(X)$. The holomorphic 2-form (unique up to scale)

$$\sigma_S \in H^0(\Omega_S^2) \cong HH_2(S) \cong \mathbb{C}$$

[22] For $H^1(T_Y) \subset HH^2(Y)$ the $\mathrm{td}^{-1/2}$ twisting makes no difference anyway: the only correction would come from the contraction of $\mathrm{td}_1^{-1/2} = -c_1(Y)/4$ with $H^1(T_Y)$. The result, in $H^2(\mathcal{O}_Y)$, is the derivative of the $(0,2)$-Hodge piece of $c_1(\omega_Y)/4$ as $Y$ is deformed by a $H^1(T_Y)$ class. This is of course zero.

generates $HH_2(S)$. Similarly let

$$\sigma_X \in H^1(\Omega^3_X) \cong HH_2(X) \cong \mathbb{C}$$

be its image under $\Phi_P^{H^*}$ (or equivalently $\Phi_P^{HH_*}$) – also a generator since $\Phi_P^{H^*}$ is injective. Using these we can see when a first order commutative deformation of $D(X)$ induces a commutative deformation of $D(S)$ entirely in terms of Hodge theory.

**Proposition 6.2.** *Let $\kappa_X \in H^1(T_X) \subset HH^2(X)$ be a first order deformation of the cubic 4-fold $X$, defining a cohomology class*

$$\tilde{\kappa}_X := \kappa_X \lrcorner\, \sigma_X \in H^{2,2}(X) \subset HH_0(X).$$

*Suppose that $\tilde{\kappa}_X = \Phi_P^{H^*}(\tilde{\kappa}_S)$ for some $(1,1)$ cohomology class $\tilde{\kappa}_S \in H^{1,1}(S) \subset HH_0(S)$. Writing $\tilde{\kappa}_S = \kappa_S \lrcorner\, \sigma_S$ for $\kappa_S \in H^1(T_S)$, we have*

$$\Phi_P^{HH^*}(\kappa_X) = \kappa_S.$$

*Proof.* We follow $\sigma_S \in H^{2,0}(S) \subset HH_2(S)$ clockwise around the commutative diagram (6.9) from the top left hand corner:

$$\begin{tikzcd} HH_2(S) \arrow[r, "\Phi_P^{HH_*}"] \arrow[d, "\Phi_P^{HH^*}(\kappa_X)"'] & HH_2(X) \arrow[d, "\kappa_X"] \\ HH_0(S) \arrow[r, "\Phi_P^{HH_*}"] & HH_0(X). \end{tikzcd}$$

It maps to $\sigma_X$ in the top right, then $\tilde{\kappa}_X$ in the bottom right. By assumption this is $\Phi_P^{HH_*}(\tilde{\kappa}_S)$, and $\Phi_P^{HH_*}$ is an injection, so on the left hand side we find that $\Phi_P^{HH^*}(\kappa_X) \lrcorner\, \sigma_S = \kappa_S \lrcorner\, \sigma_S$.

Since contraction with $\sigma_S$ gives an isomorphism[23] $HH^*(S) \to HH_{2-*}(S)$, we deduce the result claimed. ∎

## 7. Deformations

7.1. **First order.** We now have a Hodge-theoretic criterion for a commutative deformation $\kappa_X \in H^1(T_X)$ to define via $\Phi_P^{HH^*}$ a commutative deformation $\kappa_S \in H^1(T_S)$ of $S$. In this section we will show that to first order, if we deform $X$ and $S$ by the Kodaira-Spencer classes $\kappa_X$ and $\kappa_S$ respectively, the fully faithful Fourier-Mukai kernel $P \in D(S \times X)$ deforms with them. Let $A_n$ denote $\operatorname{Spec} \mathbb{C}[t]/(t^{n+1})$.

**Theorem 7.1.** *Suppose that $\kappa_X \in H^1(T_X) \subset HH^2(X)$ maps via $\Phi_P^{HH^*}$ to $\kappa_S \in H^1(T_S) \subset HH^2(S)$. Let $X_1 \to A_1$ and $S_1 \to A_1$ be the corresponding flat deformations of $X$ and $S$. Then there is a perfect complex*

$$P_1 \in D(S_1 \times_{A_1} X_1)$$

*whose derived restriction to $S \times X$ is $P$.*

[23] Via the modified HKR isomorphism this is just the isomorphism $H^i(\Lambda^j T_S) \to H^i(\Omega_S^{2-j})$ induced by $\lrcorner\,\sigma_S : \Lambda^j T_S \xrightarrow{\sim} \Omega_S^{2-j}$.

By [24, Thm. 3.1], [25, Cor. 3.4] we only have to prove the vanishing of the obstruction class

$$(\kappa_S, \kappa_X) \circ \mathrm{At}(P) \ \in \ \mathrm{Ext}^2_{S\times X}(P, P). \tag{7.1}$$

Here

$$\mathrm{At}(P) : P \longrightarrow P \otimes \Omega_{S\times X}[1] \tag{7.2}$$

is the Atiyah class of $P$. The key to the proof is the interaction of various Atiyah classes with Fourier-Mukai transforms; this is what we turn to in the next section. We think of the Kodaira-Spencer class $(\kappa_S, \kappa_X) \in H^1(T_{S\times X})$ as a morphism

$$\Omega_{S\times X} \xrightarrow{(\kappa_S,\kappa_X)} \mathcal{O}_{S\times X}[1]$$

which we compose with the Atiyah class (7.2) to give the obstruction morphism $P \to P[2]$.

**Atiyah classes.** Fix a complex $F \in D(Y)$ on a smooth projective variety $Y$. Recall the first jet space $J^1(F)$ of $F$ is defined by

$$J^1(F) := \pi_{2*}\big(\pi_1^* F \otimes \mathcal{O}_{2\Delta_Y}\big). \tag{7.3}$$

Here $\pi_i$ is projection onto the $i$th factor of $Y \times Y$, and $2\Delta_Y \subset Y \times Y$ is the subscheme whose ideal sheaf is the square $\mathcal{I}^2_{\Delta_Y}$ of the ideal sheaf of the diagonal. In other words $J^1(F)$ is the image of $F$ under the Fourier-Mukai transform $D(Y) \to D(Y)$ with kernel $\mathcal{O}_{2\Delta_Y}$. The obvious exact sequence of kernels[24]

$$0 \longrightarrow \Delta_{Y*}\Omega_Y \longrightarrow \mathcal{O}_{2\Delta_Y} \longrightarrow \mathcal{O}_{\Delta_Y} \longrightarrow 0, \tag{7.4}$$

applied to $F$ gives the standard exact triangle

$$F \otimes \Omega_Y \longrightarrow J^1(F) \longrightarrow F. \tag{7.5}$$

The Atiyah class $\mathrm{At}(F) \in \mathrm{Ext}^1(F, F \otimes \Omega_Y)$ of $F$ is defined to be the connecting homomorphism $\mathrm{At}(F)\colon F \to F \otimes \Omega_Y[1]$ (or extension class) of (7.5). For this reason, the extension class of (7.4) is called the *universal Atiyah class*

$$\mathbf{At}_Y \in \mathrm{Ext}^1_{Y\times Y}(\mathcal{O}_{\Delta_Y}, \Delta_{Y*}\Omega_Y). \tag{7.6}$$

[24]Here and below we identify the conormal bundle to $\Delta_Y$ with $\Omega_Y$ via the *first* projection $\pi_1$. This (arbitrary) choice is the origin of the signs in Corollary 7.5, which are flipped if we instead use $\pi_2$.

**Partial Atiyah classes.** When $Y = A \times B$ is a product, the Atiyah class splits,

$$\begin{aligned} \mathrm{Ext}^1(F, F \otimes \Omega_{A\times B}) \ &\cong\ \mathrm{Ext}^1(F, F\otimes\Omega_A) \oplus \mathrm{Ext}^1(F, F\otimes \Omega_B), \\ \mathrm{At}(F)\ &=\ (\mathrm{At}_A(F)\,,\ \mathrm{At}_B(F)). \end{aligned} \tag{7.7}$$

We are suppressing some obvious pullback maps for appearance's sake. We can describe the component[25] $\mathrm{At}_A(F)$ of the Atiyah class directly by noting that the exact sequence (7.4) has a natural quotient:

$$\begin{tikzcd} 0 \arrow[r] & (\Delta_{A\times B})_*\Omega_{A\times B} \arrow[r] \arrow[d] & \mathcal{O}_{2\Delta_{A\times B}} \arrow[r] \arrow[d] & \mathcal{O}_{\Delta_{A\times B}} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & (\Delta_{A\times B})_*\Omega_A \arrow[r] & \mathcal{O}_{2\Delta_A\times\Delta_B} \arrow[r] & \mathcal{O}_{\Delta_{A\times B}} \arrow[r] & 0, \end{tikzcd} \tag{7.8}$$

where $2\Delta_A \times \Delta_B$ is defined by the ideal sheaf $\mathcal{I}^2_{\Delta_A} \boxtimes \mathcal{I}_{\Delta_B}$. The lower row is the extension defined by the extension class of the top row, projected via $\Omega_{A\times B} \to \Omega_A$. Therefore it is what defines the partial Atiyah class we require. It is the sequence

$$0 \longrightarrow \Delta_{A*}\Omega_A \boxtimes \mathcal{O}_{\Delta_B} \longrightarrow \mathcal{O}_{2\Delta_A} \boxtimes \mathcal{O}_{\Delta_B} \longrightarrow \mathcal{O}_{\Delta_A} \boxtimes \mathcal{O}_{\Delta_B} \longrightarrow 0, \tag{7.9}$$

i.e. $\boxtimes\, \mathcal{O}_{\Delta_B}$ applied to the sequence (7.4) for $Y = A$. Now apply this to $F \in D(A \times B)$. We pull up to $A \times B \times A \times B$ from the first two factors, tensor with (7.9), and push down to the second two factors. This push down factors through the push down to $A \times B \times A$, which turns (7.9) into

$$0 \longrightarrow \Delta_{A*}\Omega_A \boxtimes \mathcal{O}_B \longrightarrow \mathcal{O}_{2\Delta_A} \boxtimes \mathcal{O}_B \longrightarrow \mathcal{O}_{\Delta_A} \boxtimes \mathcal{O}_B \longrightarrow 0. \tag{7.10}$$

So equivalently we can pull $F$ back to $A \times B \times A$ from the first two factors, tensor with (7.10) and push down to the last two factors. The upshot is that if we define the partial jet space by

$$J^1_A(F) := \pi_{23*}\big(\pi^*_{12}F \otimes \mathcal{O}_{2\Delta_A\times B}\big)$$

(where the projections are from $A \times B \times A$ to its factors), then $\mathrm{At}_A(F)$ is the connecting homomorphism of the exact triangle

$$F \otimes \Omega_A \longrightarrow J^1_A(F) \longrightarrow F \tag{7.11}$$

induced from (7.10).

**Lemma 7.2.** *Consider (7.4) for $Y = S$ to be an exact triangle Fourier-Mukai kernels in $D(S \times S)$:*

$$\Delta_{S*}\Omega_S \longrightarrow \mathcal{O}_{2\Delta_S} \longrightarrow \mathcal{O}_{\Delta_S}. \tag{7.12}$$

*Apply the Fourier-Mukai functor $P*$ of diagram (6.1). Then the resulting exact triangle of kernels in $D(S \times X)$ is the one (7.11) defining $\mathrm{At}_S(P)$:*

$$P \otimes \Omega_S \longrightarrow J^1_S(P) \longrightarrow P. \tag{7.13}$$

---

[25]This partial Atiyah class can also be described as the Atiyah class of $F$ *relative to* $B$, i.e. $\mathrm{At}_A(F) = \mathrm{At}_{A\times B/B}(F)$.

*Proof.* Applying $P*$ means we have to pull (7.12) back to $S \times S \times X$,

$$\Delta_{S*}\Omega_S \boxtimes \mathcal{O}_X \longrightarrow \mathcal{O}_{2\Delta_S} \boxtimes \mathcal{O}_X \longrightarrow \mathcal{O}_{\Delta_S} \boxtimes \mathcal{O}_X, \tag{7.14}$$

then tensor with $\pi_{23}^*P$ and push down by $\pi_{13*}$.

But (7.14) is precisely the sequence (7.10) for $A = S$, $B = X$ (with the second and third factors swapped). So by (7.11) we indeed get the triangle (7.13) with extension class $\mathrm{At}_S(P)$. □

Symmetrically we similarly find the following.

**Lemma 7.3.** *Consider (7.4) for $Y = X$ to be an exact triangle Fourier-Mukai kernels in $D(X \times X)$:*

$$\Delta_{X*}\Omega_X \longrightarrow \mathcal{O}_{2\Delta_X} \longrightarrow \mathcal{O}_{\Delta_X}.$$

*Apply the Fourier-Mukai functor $*P$ of diagram (6.1). Then the resulting exact triangle of kernels in $D(S \times X)$ is the one (7.11) defining $\mathrm{At}_X(P)$:*

$$P \otimes \Omega_X \longrightarrow J^1_X(P) \longrightarrow P.$$ □

**The universal Atiyah class.** The universal Atiyah class $\mathbf{At}_S$ (7.6) is *not* the Atiyah class $\mathrm{At}(\mathcal{O}_{\Delta_S})$ of $\mathcal{O}_{\Delta_S}$ on $S \times S$, though it is a component of it. To understand the relation we need the following.

**Lemma 7.4.** *Let $Z \subset Y$ be a subscheme of a smooth variety, giving the standard short exact sequence*

$$0 \longrightarrow \mathcal{I}_Z/\mathcal{I}_Z^2 \longrightarrow \mathcal{O}_{2Z} \longrightarrow \mathcal{O}_Z \longrightarrow 0. \tag{7.15}$$

*The image of its extension class in $\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{I}_Z/\mathcal{I}_Z^2)$ under the canonical map $\mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_Y|_Z$ is the Atiyah class*

$$\mathrm{At}(\mathcal{O}_Z) \in \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z \otimes \Omega_Y).$$

*Proof.* We apply $\pi_{2*}(\pi_1^*\mathcal{O}_Z \otimes \ \cdot\ )$ to the exact sequence

$$0 \longrightarrow \Omega_Y \otimes \mathcal{O}_{\Delta_Y} \longrightarrow \mathcal{O}_{2\Delta_Y} \longrightarrow \mathcal{O}_{\Delta_Y} \longrightarrow 0$$

on $Y \times Y$. Since these sheaves are $\pi_1$-flat, we can take the underived tensor product of structure sheaves, giving the structure sheaves of the intersections. The result is $\pi_{2*}$ of the exact sequence

$$0 \longrightarrow \Omega_Y \otimes \mathcal{O}_{\Delta_Z} \longrightarrow \mathcal{O}_{2(\Delta_Z \subset Z \times Y)} \longrightarrow \mathcal{O}_{\Delta_Z} \longrightarrow 0.$$

Here $2(\Delta_Z \subset Z \times Y)$ denotes the doubling of $\Delta_Z$ inside $Z \times Y$. This scheme maps onto $2Z \subset Y$ by the projection $\pi_2$. Therefore using $\pi_2^*$ to pull back functions gives a map of sheaves from $\mathcal{O}_{2Z}$ to $\mathcal{O}_{2(\Delta_Z \subset Z \times Y)}$. Of course this is *not* a map of $\mathcal{O}_{Y_1 \times Y_2}$-modules, but it *is* a map of $\mathcal{O}_{Y_2}$-modules. Therefore after we apply $\pi_{2*}$ we get a map of $\mathcal{O}_Y$-modules

$$\begin{tikzcd}
0 \arrow[r] & \Omega_Y \otimes \mathcal{O}_Z \arrow[r] & \pi_{2*}\mathcal{O}_{2(\Delta_Z \subset Z \times Y)} \arrow[r] & \mathcal{O}_Z \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{I}_Z/\mathcal{I}_Z^2 \arrow[r] & \mathcal{O}_{2Z} \arrow[r] \arrow[u] & \mathcal{O}_Z \arrow[r] & 0.
\end{tikzcd}$$

It is easy to see that this completes in the obvious way with vertical arrows on the left and right. Since the upper central term is $J^1(\mathcal{O}_Z)$ (7.3) and the upper row is the sequence (7.5) defining $\mathrm{At}(\mathcal{O}_Z)$, this gives the result claimed. $\square$

We apply this to $Z = \Delta_S$ inside $Y = S \times S$, for which the canonical map $\mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_Y|_Z$ splits:

$$\Omega_{S\times S}|_{\Delta_S} \;\cong N^*_{\Delta_S} \oplus \Omega_{\Delta_S}. \tag{7.16}$$

Here $N^*_{\Delta_S}$ denotes the conormal bundle. Furthermore, the sequence (7.15) becomes (7.4), with extension class the universal Atiyah class $\mathbf{At}_S$ of (7.6). Therefore $\mathrm{At}(\mathcal{O}_{\Delta_S})$ splits too, with respect to (7.16), as

$$\mathrm{At}(\mathcal{O}_{\Delta_S}) = \mathbf{At}_S \oplus 0 \;\in\; \mathrm{Ext}^1\big(\mathcal{O}_{\Delta_S}, \mathcal{O}_{\Delta_S} \otimes (N^*_S \oplus \Omega_{\Delta_S})\big). \tag{7.17}$$

In particular we have proved the following.

**Corollary 7.5.** *Denote the two partial Atiyah classes (7.7) of $\mathcal{O}_{\Delta_S}$ by $\mathrm{At}_1(\mathcal{O}_{\Delta_S})$ and $\mathrm{At}_2(\mathcal{O}_{\Delta_S})$. Then*

$$\mathrm{At}_1(\mathcal{O}_{\Delta_S}) = \mathbf{At}_S = -\,\mathrm{At}_2(\mathcal{O}_{\Delta_S}) \;\in\; \mathrm{Ext}^1(\mathcal{O}_{\Delta_S}, \Omega_{\Delta_S}).$$

*Proof of Theorem 7.1.* Following Toda [42], the idea of the proof is to see the HKR isomorphism as identifying: (1) first order deformations of $D(S)$, with (2) the corresponding obstructions to deforming the identity Fourier-Mukai kernel $\mathcal{O}_{\Delta_S} \in D(S \times S)$ as the first $S$ (or $D(S)$) factor deforms and the second remains fixed. Applying that same philosophy to $P \in D(S \times X)$ as well quickly gives the result.

The Kodaira-Spencer class $\kappa_S \in H^1(T_S)$ induces a corresponding first order deformation $\pi_1^*\kappa_S \in H^1(T_{S\times S})$ of $S \times S$ (deforming the first factor and fixing the second). The obstruction to deforming $\mathcal{O}_{\Delta_S} \in D(S \times S)$ with this deformation is [24, 25]

$$\pi_1^*\kappa_S \circ \mathrm{At}(\mathcal{O}_{\Delta_S}) \in \mathrm{Ext}^2(\mathcal{O}_{\Delta_S}, \mathcal{O}_{\Delta_S}) = HH^2(S).$$

By (7.17) and Corollary 7.5 this equals

$$\pi_1^*\kappa_S \circ \mathrm{At}_1(\mathcal{O}_{\Delta_S}) = \pi_1^*\kappa_S \circ \mathbf{At}_S, \tag{7.18}$$

which is precisely the image of $\kappa_S$ under the HKR isomorphism[26] (6.10).

Similarly, the image of $\kappa_X \in H^1(T_X)$ in $\mathrm{Ext}^2(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) = HH^2(X)$ is given by

$$\pi_1^*\kappa_X \circ \mathrm{At}_1(\mathcal{O}_{\Delta_X}) = -\pi_2^*\kappa_X \circ \mathrm{At}_2(\mathcal{O}_{\Delta_X})$$

by Corollary 7.5. To describe its image under $\Phi_P^{HH^*}$ (6.3) we have to chase around the diagram (6.1). Under the Fourier-Mukai composition $*P$ of that diagram it maps to

$$-\,\pi_2^*\kappa_X \circ \mathrm{At}_X(P) \in \mathrm{Ext}^2_{S\times X}(P, P) \tag{7.19}$$

[26] The HKR isomorphism is most easily described using the exponential of $\mathbf{At}_S$ [11, 16], but in this low degree this reduces to $\circ\mathbf{At}_S : H^1(T_S) \hookrightarrow \mathrm{Ext}^2(\mathcal{O}_{\Delta_S}, \mathcal{O}_{\Delta_S}) = HH^2(S)$. As noted in Footnote 22 the $\mathrm{td}^{-1/2}$ twisting acts as the identity on $H^1(T)$ classes.

by Lemma 7.3.

Similarly, under the Fourier-Mukai composition $P*$ of diagram (6.1), $\kappa_S$ (or rather its image (7.18) in $HH^2(S)$) is taken to

$$\pi_1^*\kappa_S \circ \mathrm{At}_S(P) \in \mathrm{Ext}^2_{S\times X}(P,P) \tag{7.20}$$

by Lemma 7.2.

Since $\Phi_P^{HH^*}(\kappa_X) = \kappa_S$, the classes (7.20, 7.19) are equal. In particular,

$$(\kappa_S, \kappa_X) \circ \mathrm{At}(P) = \pi_1^*\kappa_S \circ \mathrm{At}_S(P) + \pi_2^*\kappa_X \circ \mathrm{At}_X(P) = 0$$

in $\mathrm{Ext}^2(P,P)$. Thus the obstruction class (7.1) vanishes as required. $\square$

7.2. **All orders by $T^1$-lifting.** We now extend Theorem (7.1) to all orders using $T^1$-lifting methods [40, 26]. The philosophy is that, to extend an equivalence from a family over $A_n$ to one over $A_{n+1}$, it is enough to understand deformations "sideways" from $A_n$ to $A_n \times A_1$. For this latter problem we can invoke Theorem 7.1 on deforming over $A_1$.

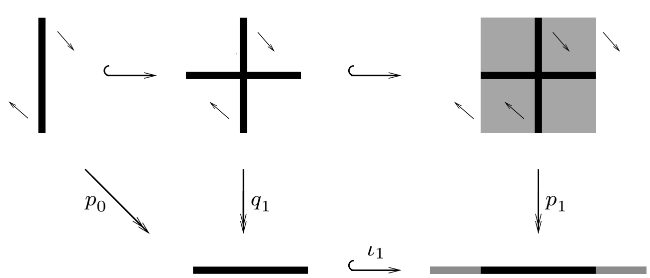
We use the standard Artinian spaces

$$\begin{aligned} A_n &:= \operatorname{Spec} \mathbb{C}[t]/(t^{n+1}), \\ B_n &:= A_n \times A_1 = \operatorname{Spec} \mathbb{C}[x,y]/(x^{n+1}, y^2), \\ C_n &:= \operatorname{Spec} \mathbb{C}[x,y]/(x^{n+1}, xy^n, y^2). \end{aligned}$$

The key to $T^1$-lifting is the following maps between them:

$$\begin{tikzcd} B_{n-1} \arrow[r, hook] \arrow[dr, "p_{n-1}"'] & C_n \arrow[r, hook] \arrow[d, "q_n"] & B_n \arrow[d, "p_n"] & x+y \\ & A_n \arrow[r, hook, "\iota_n"] & A_{n+1}, & t. \arrow[u, mapsto] \end{tikzcd} \tag{7.21}$$

The surjections $p_{n-1}, q_n$ and $p_n$ are defined by their action on $t$, which they pull back to $x+y$. So they act "diagonally", mixing up the $A_n$ and $A_1$ axes; we picture them as follows (for $n=1$).




Since $p_n^*\left(\frac{t^{n+1}}{n+1}\right) = x^n y$, the ideal $(t^{n+1})$ of $A_n \subset A_{n+1}$ is isomorphic, via the pull back $p_n^*$, to the ideal[27] $(x^n y)$ of $C_n \subset B_n$. Therefore we will find

[27]These two ideals "correspond" to the grey shaded areas in the figure.

that *extending a deformation from $C_n$ to $B_n$ becomes the same problem as extending from $A_n$ to $A_{n+1}$*. In fact we have the diagram of exact sequences

$$\begin{tikzcd} (t^{n+1}) \arrow[r] \arrow[d, equal] & \mathcal{O}_{A_{n+1}} \arrow[r] \arrow[d, "p_n^*"] & \mathcal{O}_{A_n} \arrow[d, "q_n^*"] \\ p_{n*}(x^n y) \arrow[r] & p_{n*}\mathcal{O}_{B_n} \arrow[r] \arrow[d] & q_{n*}\mathcal{O}_{C_n} \arrow[d] \\ & Q_n \arrow[r, equal] & Q_n \end{tikzcd} \tag{7.22}$$

on $A_{n+1}$. Here the right hand vertical maps are $\mathcal{O}_{A_{n+1}} \to p_{n*}p_n^*\mathcal{O}_{A_{n+1}}$ and $\mathcal{O}_{A_n} \to q_{n*}q_n^*\mathcal{O}_{A_n}$ respectively. We denote the cokernel of the latter by $Q_n$, and suppress pushforwards by obvious inclusions. We consider (7.22) as showing how to recover the flat deformation $\mathcal{O}_{A_{n+1}}$ over $A_{n+1}$ of $\mathcal{O}_{A_n}$ over $A_n$ from the flat deformation $\mathcal{O}_{B_n}$ of $\mathcal{O}_{C_n}$: i.e. push down and take the kernel of the map to $Q_n$.

The case of deformations of complexes of sheaves is not much harder. Suppose we are given a smooth projective family $\mathcal{A}_{n+1} \to A_{n+1}$. Base-changing by $\iota_n$ gives a family $\mathcal{A}_n \to A_n$. Similarly base-changing by $p_n$ and $q_n$ gives $\mathcal{B}_n \to B_n$ and $\mathcal{C}_n \to C_n$ respectively. We use the same symbols $p_n, q_n, \iota_n$ to denote the induced maps between them.

We start with a perfect complex $P_n \in D(\mathcal{A}_n)$, whose (derived) restriction to $A_{n-1}$ we denote $P_{n-1} := \iota_{n-1}^* P_n$. To extend $P_n$ to some $P_{n+1}$ on $\mathcal{A}_{n+1}$, we consider sideways first order deformations of $P_n$ over $A_n \times A_1 = B_n$, i.e. perfect complexes on $\mathcal{B}_n$ whose base-change by $A_n \times \{0\} \hookrightarrow A_n \times A_1$ is $P_n$.

**Proposition 7.6** ($T^1$-lifting for complexes of sheaves[28])**.** *Suppose there exists a first order deformation*

$$\widetilde{P}_{n+1} \in D(\mathcal{B}_n)$$

*of $P_n \in D(\mathcal{A}_n)$ whose restriction to $\mathcal{B}_{n-1}$ is*

$$p_{n-1}^*(P_n) \in D\big(\mathcal{B}_{n-1}\big). \tag{7.23}$$

*Then there exists $P_{n+1} \in D(\mathcal{A}_{n+1})$ whose restriction to $\mathcal{A}_n$ is $P_n$ (and such that $p_n^* P_{n+1} = \widetilde{P}_{n+1}$).*

*Proof.* By assumption, $\widetilde{P}_{n+1}$ restricts to $P_n$ on $\mathcal{A}_n$ and to $p_{n-1}^* P_n$ on $\mathcal{B}_{n-1}$, both of which restrict to $P_{n-1}$ on $\mathcal{A}_{n-1}$. But $\mathcal{C}_n = \mathcal{B}_{n-1} \cup_{\mathcal{A}_{n-1}} \mathcal{A}_n$, so the restriction of $\widetilde{P}_{n+1}$ to $\mathcal{C}_n$ is $q_n^* P_n$. Suppressing pushforwards by some obvious inclusions, we get the exact triangle

$$P_0 \otimes (x^n y) \longrightarrow \widetilde{P}_{n+1} \longrightarrow q_n^* P_n$$

[28] The conditions of this Proposition can be stated more attractively when the family $\mathcal{A}_n$ is a product $Y \times A_n$, i.e. when we are not deforming the underlying variety $Y$. We ask for a deformation class $e_{n+1} \in \operatorname{Ext}^1_{\mathcal{A}_n}(P_n, P_n)$ whose restriction to $\mathcal{A}_{n-1}$ gives the extension class $e_n \in \operatorname{Ext}^1_{\mathcal{A}_{n-1}}(P_{n-1}, P_{n-1})$ of $p_{n-1}^*(P_n)$. (The complex $p_{n-1}^*(P_n)$ on $\mathcal{B}_{n-1}$ is a first order deformation of $P_{n-1}$ since that is what it restricts to on $\mathcal{A}_{n-1}$.)

on $\mathcal{B}_n$. Now mimic diagram (7.22). Pushing down by $p_{n*}$ gives the middle row of the following diagram of exact triangles on $\mathcal{A}_{n+1}$:

$$\begin{tikzcd} P_0 \otimes (t^{n+1}) \arrow[r] \arrow[d, equal] & P_{n+1} \arrow[r] \arrow[d] & P_n \arrow[d, "q_n^*"] \\ P_0 \otimes p_{n*}(x^n y) \arrow[r] & p_{n*}\widetilde{P}_{n+1} \arrow[r] \arrow[d] & q_{n*}q_n^* P_n \arrow[d] \\ & Q_n \arrow[r, equal] & Q_n. \end{tikzcd} \tag{7.24}$$

The right hand column defines $Q_n := \mathrm{Cone}\,(P_n \to q_{n*}q_n^*P_n)$, then we define $P_{n+1} := \mathrm{Cone}\,(p_{n*}\widetilde{P}_{n+1} \to Q_n)[-1]$ by the middle column.

The arrows $P_{n+1} \to \iota_{n*}P_n$ and $P_{n+1} \to p_{n*}\widetilde{P}_{n+1}$ in (7.24) induce, by adjunction,

$$\iota_n^* P_{n+1} \longrightarrow P_n \qquad \text{and} \qquad p_n^* P_{n+1} \longrightarrow \widetilde{P}_{n+1}. \tag{7.25}$$

We claim that these are quasi-isomorphisms, which we can check locally as follows. Since $\widetilde{P}_{n+1}$ is perfect we may take it to be a finite complex of locally free sheaves whose restriction to $\mathcal{A}_n$ is a finite complex of locally frees representing $P_n$. Locally free sheaves are adapted to all of the pullback and (finite!) pushforward functors in (7.24): they can be applied term by term to the sheaves in the complex to produce the derived functors. Working locally then, we need only check that the maps (7.25) are isomorphisms for free sheaves, which follows from (7.22).

Finally we claim that $P_{n+1}$ is perfect. By the first isomorphism of (7.25), the (derived) restriction of $P_{n+1}$ to any point of $\mathcal{A}_0$ has cohomology in only finitely many degrees (because $P_n$ is perfect). Therefore, by the Nakayama lemma, an infinite locally free resolution of $P_{n+1}$ can be trimmed to give a finite one. $\square$

In our application we take any smooth curve through the point $0 \in \mathcal{C}_d^{\mathrm{lev}}$ of Section 5.2, and complete at 0. Thus we have smooth families $\mathcal{S}$ and $\mathcal{X}$ of K3 surfaces and cubic 4-folds respectively, over the formal curve $A_\infty := \mathrm{Spec}\,\mathbb{C}[\![t]\!]$. We let their restrictions to $A_n \subset A_\infty$ be denoted by $S_n$, $X_n$. Over the central fibres $S_0$ and $X_0$ we have the Fourier-Mukai kernel

$$P_0 \in D(S_0 \times X_0)$$

of a *fully faithful* embedding $D(S_0) \hookrightarrow D(X_0)$, inducing an embedding

$$\Phi_{P_0}^{H^*} \colon H^*(S_0) \lhook\joinrel\longrightarrow H^*(X_0).$$

Using the natural trivialisations [9, Prop. 3.8]

$$H^*_{dR}(\mathcal{S}/A_\infty) \cong H^*(S_0) \otimes \mathbb{C}[\![t]\!], \qquad H^*_{dR}(\mathcal{X}/A_\infty) \cong H^*(X_0) \otimes \mathbb{C}[\![t]\!], \tag{7.26}$$

we get the inclusion

$$H^*_{dR}(\mathcal{S}/A_\infty) \lhook\joinrel\longrightarrow H^*_{dR}(\mathcal{X}/A_\infty). \tag{7.27}$$

Up to units in $\mathbb{C}[\![t]\!]$ there is a unique holomorphic 2-form on the fibres of $\mathcal{S}$ and a unique $(3,1)$-form on the fibres of $\mathcal{X}$:

$$\sigma_S \in H^2_{dR}(\mathcal{S}/A_\infty), \qquad \sigma_X \in H^4_{dR}(\mathcal{X}/A_\infty).$$

By Theorem 5.3, the span of $\sigma_S$ is taken to the span of $\sigma_X$ by $\Phi^{H^*}_{P_0}$. Rescaling $\sigma_X$ if necessary, we are in the following situation.

**Theorem 7.7.** *Suppose that* $\Phi^{H^*}_{P_0}$ *maps*

$$\sigma_S \longmapsto \sigma_X. \tag{7.28}$$

*Then* $P_0$ *extends to all orders. That is, there exist* $P_n \in D(S_n \times_{A_n} X_n)$ *satisfying* $\iota^*_{n-1} P_n \cong P_{n-1}$ *for all* $n \geq 1$*, defining full and faithful embeddings* $\Phi_{P_n} \colon D(S_n) \to \mathcal{A}_{X_n} \subset D(X_n)$.

*Proof.* We will apply Proposition 7.6 on the spaces

$$\mathcal{A}_n := \mathcal{S} \times_{A_n} \mathcal{X} \longrightarrow A_n.$$

Base-changing by the maps (7.21) gives similar families $\mathcal{C}_n \to C_n$ and $\mathcal{B}_n \to B_n$. (Here we also allow $n = \infty$.)

By induction we suppose we have a perfect complex $P_n \in D(\mathcal{A}_n)$ inducing a fully faithful embedding $D(S_n) \hookrightarrow D(X_n)$. This is certainly true in the base case $n = 0$. To produce $P_{n+1}$ we now proceed exactly as in Sections 6 and 7.1, except now relative to the base $A_n$ in place of $A_0 = \operatorname{Spec} \mathbb{C}$.

Considering $\mathcal{B}_n \to A_n \times A_1$ to be a first order deformation (in the category of schemes over $A_n$) of $\mathcal{A}_n \to A_n$, it is characterised by its Kodaira-Spencer class

$$\kappa_n \in H^1(T_{\mathcal{A}_n/A_n}).$$

With respect to the splitting $\mathcal{A}_n = S_n \times_{A_n} X_n$, this is

$$\kappa_n = (\kappa_{S_n}, \kappa_{X_n}) \in H^1\big(T_{S_n/A_n} \oplus T_{X_n/A_n}\big).$$

Contracting with $\sigma_S|_{S_n} \oplus \sigma_X|_{X_n}$ gives the relative cohomology class

$$(\tilde\kappa_{S_n}, \tilde\kappa_{X_n}) \in H^1\big(\Omega^1_{S_n/A_n}\big) \oplus H^2\big(\Omega^2_{X_n/A_n}\big).$$

Differentiating (7.28) with respect to $t$ we find that $\Phi^{H^*}_{P_0}$ maps

$$H^*(S_0) \otimes \mathbb{C}[\![t]\!] \ \ni\ \dot\sigma_S \longmapsto \dot\sigma_X \ \in\ H^*(X_0) \otimes \mathbb{C}[\![t]\!].$$

By Griffiths' classic variation of Hodge structure calculation[29], this gives

$$H^*(S_0) \otimes \mathbb{C}[\![t]\!] \ \ni\ \tilde\kappa_{S_\infty} \longmapsto \tilde\kappa_{X_\infty} \ \in\ H^*(X_0) \otimes \mathbb{C}[\![t]\!].$$

[29]Griffiths shows that if $\sigma_t \in H^{p,q}(Y_t)$ $\forall t$ then $[\dot\sigma_t]^{p-1,q+1} = \kappa_t \lrcorner\, \sigma_t \in H^{p-1,q+1}(Y_t)$, where $\kappa_t \in H^1(T_{Y_t})$ is the Kodaira-Spencer class, and we differentiate in the trivialisation (7.26) – i.e. with respect to the Gauss-Manin connection. Projecting $\dot\sigma_S$ to its $(1,1)$ component, and $\dot\sigma_X$ to its $(2,2)$ component, removes multiples of $\sigma_S$ and $\sigma_X$ respectively. Since these are mapped to each other by (7.28), this means the projections are too.

Base-changing back to $A_n$ we find that (7.27) takes $\tilde\kappa_{S_n}$ to $\tilde\kappa_{X_n}$. Thus by Proposition 6.2 over the base $A_n$, the map on Hochschild cohomology induced by $P_n$ takes $\kappa_{X_n}$ to $\kappa_{S_n}$:

$$\begin{array}{ccc} HH^2(\mathcal{X}_n/A_n) & \longrightarrow & HH^2(\mathcal{S}_n/A_n) \\ \kappa_{X_n} & \longmapsto & \kappa_{S_n}. \end{array}$$

Thus by Theorem 7.1 applied to $P_n \in D(\mathcal{A}_n)$ over the base $A_n$ we find that there is a perfect complex $\widetilde{P}_{n+1} \in D(\mathcal{B}_n)$ whose restriction to $\mathcal{A}_n$ is $P_n$. It is unique since

$$\mathrm{Ext}^1_{\mathcal{A}_n}(P_n, P_n) \stackrel{(6.2)}{=} HH^1(S_n/A_n) \stackrel{(6.10)}{=} H^1(\mathcal{O}_{S_n/A_n}) \oplus H^0(T_{S_n/A_n})$$

vanishes for $S_n/A_n$ a family of K3 surfaces. By uniqueness its restriction to $\mathcal{B}_{n-1}$ is therefore $p_{n-1}^* P_n$, so we can apply Proposition 7.6 to produce $P_{n+1} \in D(\mathcal{A}_{n+1})$.

The Fourier-Mukai composition of the kernel $P_{n+1}$ with the kernel (6.6) of its right adjoint gives a perfect complex on $S_{n+1} \times_{A_{n+1}} S_{n+1}$ whose restriction to $S_0 \times S_0$ is $\mathcal{O}_{\Delta_{S_0}}$. But $\mathcal{O}_{\Delta_{S_0}}$ is rigid: $\mathrm{Ext}^1_{S_0\times S_0}(\mathcal{O}_{\Delta_{S_0}}, \mathcal{O}_{\Delta_{S_0}}) = 0$. Thus the composition is $\mathcal{O}_{\Delta_{S_{n+1}}}$, so $\Phi_{P_{n+1}}$ is full and faithful. Its image is $\mathcal{A}_{X_{n+1}} \subset D(X_{n+1})$ because being orthogonal to the pullbacks from $X$ of $\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)$ is also an open condition by semi-continuity. □

Thus the kernel $P_0$ given to us by Theorem 4.1 and Proposition 5.2 deforms to all orders along any smooth curve in the smooth space $\mathcal{C}_d^{\mathrm{lev}}$. It follows by standard deformation theory that its deformations are unobstructed, and it deforms to all orders over $\mathcal{C}_d^{\mathrm{lev}}$ itself. Therefore by [31, Prop. 3.6.1] the kernel $P_0$ in fact deforms over the formal neighbourhood $\widehat{Z} = \operatorname{Spec} \widehat{\mathcal{O}}_{\mathcal{C}_d^{\mathrm{lev}},0}$ of our point $0 \in \mathcal{C}_d^{\mathrm{lev}}$ to give a kernel in $D(\mathcal{S} \times_{\widehat{Z}} \mathcal{X})$. Since having no negative self-Exts is an open condition satisfied by $P_0$ (6.2), this gives a formal point $\widehat{Z}$ in the stack of complexes with no negative self-Exts on the fibres of $\mathcal{S} \times_{\widehat{Z}} \mathcal{X} \to \widehat{Z}$.

Lieblich [31] shows this is an Artin stack of local finite presentation, so $\widehat{Z}$ lies in a smooth scheme $Z$ over $\mathcal{C}_d^{\mathrm{lev}}$ in the stack. Passing to a Zariski open in $Z$ if necessary we ensure the projection $Z \to \mathcal{C}_d^{\mathrm{lev}}$ is an embedding (as it is at $\widehat{Z}$).

The universal complex pulls back to a twisted complex $P$ on $\mathcal{S} \times_Z \mathcal{X}$ whose restriction to any fibre $S \times X$ is an untwisted complex. As at the end of the proof of Theorem 7.7, defining an equivalence $D(S) \to \mathcal{A}_X$ is an open condition satisfied on the central fibre $S_0 \times X_0$.

Therefore, after shrinking $Z \subset \mathcal{C}_d^{\mathrm{lev}}$ if necessary, we get an equivalence $D(S) \to \mathcal{A}_X$ for all $X \in Z$. Projecting $Z$ by the finite map to $\mathcal{C}_d$ gives the Zariski open subset claimed in Theorem 1.1.

7.3. **Outlook.** We consider the problem of extending Theorem 1.1 to the whole of $\mathcal{C}_d$ to be a purely technical problem, but one for which current techniques seem inadequate. The problem is to produce from a family of equivalences $\Phi_t\colon D(S_t) \to \mathcal{A}_{X_t}$ a limiting equivalence $\Phi_\infty\colon D(S_\infty) \to \mathcal{A}_{X_\infty}$.

The quickest route would be to use stability conditions of some kind, if they were better understood. Ideally we would see $S_t$ as a moduli space of *stable* objects in $\mathcal{A}_{X_t} \subset D(X_t)$ and pass to the limit $t = \infty$ this way. Alternatively we would like a stability condition on $S_t \times X_t$ in which the Fourier-Mukai kernel $P_t$ of $\Phi_t$ is stable, and take the limit of $P_t$ as a stable object of $D(S_\infty \times X_\infty)$.

An alternative is to switch points of view from considering $S_t$ to be a moduli space of objects in $\mathcal{A}_{X_t} \subset D(X_t)$ and instead consider $X_t$ as a moduli space of objects on $S_t$. Projecting $\mathcal{O}_x,\ x \in X_t$, into $\mathcal{A}_{X_t}$ gives an object $P_x$ which sits inside an 8-dimensional holomorphic symplectic moduli space $M_t$ of objects of $\mathcal{A}_{X_t} \subset D(X_t)$, studied in forthcoming work of Lehn et al. [30]. Considered via the equivalence as a moduli space of objects on the K3 surface $S_t$ we call it $N_t \cong M_t$. The map

$$f_t\colon X_t \longrightarrow N_t, \qquad x \mapsto P_x,$$

expresses $X_t$ as a complex Lagrangian submanifold of $N_t$. Since K3s are so well understood, taking the limit moduli space $N_\infty$ of stable objects on $S_\infty$ should be no problem, so we would like to take the limit of the maps $f_t$.

*A priori* this might involve blowing up $X_\infty$ so instead we might try to take the limit of the isomorphisms $M_t \cong N_t$, at least near $X_t \subset M_t$. Namely, replacing $M_t$ by a Zariski open neighbourhood of $X_t \subset M_t$ gives a Zariski open in the Artin stack of *all* objects (with no negative self-Exts) of $D(X_t)$. This stack *does* behave well in families [31], giving a limiting stack of objects in $D(X_\infty)$. These include $P_x,\ x \in X_\infty$, so this stack contains the scheme $X_\infty$, and we can set $M_\infty$ to be a Zariski open neighbourhood of this. So we have two families of quasi-projective *holomorphic symplectic* varieties $M_t,\ N_t,\ t \in \mathbb{C} \cup \{\infty\}$, isomorphic away from $t = \infty$. We expect that this gives a birational equivalence between $M_\infty$ and $N_\infty$, and so a derived equivalence between compactly supported derived categories too. The upshot should be a kernel on $S_\infty \times X_\infty$ which we would expect to define a fully faithful embedding. There is clearly a lot of work to do here.

Finally we could instead try to use $F(X_t)$ [8], the Fano variety of lines on $X_t$, which is also a moduli space of objects in $\mathcal{A}_{X_t}$ (namely the projection into $\mathcal{A}_{X_t}$ of the structure sheaves of the lines [28, §5]). The equivalence therefore makes it isomorphic to a 4-dimensional holomorphic symplectic moduli space $M_t$ of objects in $D(S)$. In the limit the isomorphism $F(X_t) \cong M_t$ makes $F(X_\infty)$ and $M_\infty$ birational and so derived equivalent by work of Namikawa.

Then consider the composition

$$D(S_\infty) \longrightarrow D(M_\infty) \longrightarrow D(F(X_\infty)) \longrightarrow \mathcal{A}_{X_\infty} \subset D(X_\infty), \tag{7.29}$$

where the first arrow is given by the universal complex, the second is the derived equivalence, and the third uses the universal complex that sees $F(X_\infty)$ as a moduli space of objects in $\mathcal{A}_{X_\infty}$. We expect this composition to be of the form $\Phi_\infty \oplus \Phi_\infty[-2]$, where $\Phi_\infty$ is the equivalence we seek.

One way to try to prove this is as follows. Consider the (adjoint of) the composition of the first two arrows as expressing $F(X_\infty)$ as a moduli space of objects on $S_\infty$. One should check that they are *simple* objects. They form a spanning set, so it should be enough to see that the composition (7.29) acts as a direct sum $\Phi_\infty \oplus \Phi_\infty[-2]$ of an equivalence and its shift on these objects. So we reduce to studying the composition

$$D(F(X_\infty)) \longrightarrow D(S_\infty) \longrightarrow D(F(X_\infty)) \longrightarrow \mathcal{A}_{X_\infty} \subset D(X_\infty).$$

Now the composition of the first two arrows is the endofunctor of $D(F(X_\infty))$ studied by Markman and Mehrotra in [35]. Though the definition of this endofunctor involves seeing $F(X_\infty)$ as a moduli space of objects on $S_\infty$, they expect it to be independent of this description, and to be canonically associated to the holomorphic symplectic manifold $F(X_\infty)$. So we would get the same endofunctor by thinking of $F(X_\infty)$ as a moduli space of objects of $\mathcal{A}_X$ via the (adjoint of) the third arrow above, and the composition becomes

$$D(F(X_\infty)) \longrightarrow \mathcal{A}_{X_\infty} \longrightarrow D(F(X_\infty)) \longrightarrow \mathcal{A}_{X_\infty} \subset D(X_\infty).$$

But by [1, Thm. 4], the composition of the second and third arrows in this last sequence is $\mathrm{id} \oplus [-2]$, as required.

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